Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Find the value of x, if $[1 \quad x \quad 1]\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0.$

Ans.

Given, $[1 \quad x \quad 1]_{1\times3}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}_{3\times3}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}_{3\times1} = 0$

$\Rightarrow \quad [1+2x+15 \quad 3+5x+3 \quad 2+x+2]\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$

$\Rightarrow \quad [16+2x \quad 6+5x \quad 4+x]\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$

$\Rightarrow \quad [(16+2x).1 + (6+5x).2 + (4+x).x] = 0$

$\Rightarrow \quad (16+2x) + (12+10x) + (4x+x2) = 0$

$\Rightarrow \quad x^2 + 16x + 28 = 0 \quad \Rightarrow \quad (x+14)(x+2) = 0$

$\Rightarrow \quad x + 14 = 0 \quad$ or $\quad x + 2 = 0$

Hence, $\quad x = -14 \quad$ or $\quad x = -2$

Q.2. For the following matrices *A* and *B*, verify that (*AB*)' = *B*'*A*'.

$A = \begin{bmatrix}1\\ -4\\ 3\end{bmatrix}, B = (-1, 2, 1)$

Ans.

Given: $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \quad B = (-1,\ 2,\ 1)$

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} [-1\ \ 2\ \ 1] = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

$$B'A' = (-1\ \ 2\ \ 1)' \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [1\ \ -4\ \ 3] = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

$\therefore \quad (AB)' = B'A'.$

Q.3. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then find the values of *a* and *b*.

Ans.

Here, $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$$\therefore \quad A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$\Rightarrow \quad (A+B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} \cdot \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} = \begin{bmatrix} 1 + a^2 + 2a & 0 \\ 2 + 2a + b + ab - 4 - 2b & -2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + 2a + 1 & 0 \\ 2a - b + ab - 2 & 4 \end{bmatrix}$$

$$\text{Again} \quad A^2 + B^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2 + b & a - 1 \\ ab - b & b + 1 \end{bmatrix} = \begin{bmatrix} a^2 + b - 1 & a - 1 \\ ab - b & b \end{bmatrix}$$

Given, $(A + B)^2 = A^2 + B^2$

Given, $(A+B)^2 = A^2 + B^2$

$$\begin{bmatrix} a^2+2a+1 & 0 \\ 2a-b+ab-2 & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

Equating the corresponding elements, we get

$a^2 + 2a + 1 = a^2 + b - 1 \quad \Rightarrow \quad 2a - b = -2 \qquad ...(i)$

$a - 1 = 0 \quad \Rightarrow \quad a = 1 \qquad ...(ii)$

$2a - b + ab - 2 = ab - b \quad \Rightarrow \quad 2a - 2 = 0 \qquad ...(iii)$

$b = 4 \qquad ...(iv)$

$a = 1$, $b = 4$ satisfy all four equations (i), (ii), (iii) and (iv)

Hence, $a = 1$, $b = 4$.

Q.4. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix *D* such that *CD* – *AB* = *O*.

Ans.

Since *A*, *B*, *C* are all square matrices of order 2, and *CD* – *AB* is well defined, *D* must be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $CD - AB = 0$ gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or $$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equating the corresponding elements of matrices, we get

$$2a + 5c - 3 = 0 \quad ...(i)$$

$$3a + 8c - 43 = 0 \quad ...(ii)$$

$$2b + 5d = 0 \quad ...(iii)$$

and $$3b + 8d - 22 = 0 \quad ...(iv)$$

Solving (i) and (ii), we get $a = -191$, $c = 77$. Solving (iii) and (iv), we get $b = -110$, $d = 44$.

Therefore $$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Q.5. Express the following matrix as the sum of a symmetric and skew symmetric matrix, and verify your result.

$$\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Ans.

Let $A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

A can be expressed as

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A'), \quad ...(i) \quad \left[\because \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{2A}{2} = A\right]$$

where, $A + A'$ and $A - A'$ are symmetric and skew symmetric matrices respectively.

Now, $$A + A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

Putting these values in (i) we get

$$A = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}$$

Verification:

$$\Rightarrow \quad \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & \frac{1}{2} - \frac{5}{2} & -\frac{5}{2} - \frac{3}{2} \\ \frac{1}{2} + \frac{5}{2} & -2+0 & -2-3 \\ -\frac{5}{2} + \frac{3}{2} & -2+3 & 2+0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = A$$

Q.6. Show that $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ **satisfies the equation** $x^2 - 6x + 17 = 0$**. Hence, find** A^{-1}

Ans.

We have, $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

$$\therefore \quad A^2 = A.A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4-9 & -6-12 \\ 6+12 & -9+16 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$$

$$6A = 6\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} \text{ and } 17I = 17\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$\therefore \quad A^2 - 6A + 17I_2 = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} -5-12+17 & -18+18+0 \\ 18-18+0 & 7-24+17 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence, matrix A satisfies the equation, $x^2 - 6x + 17 = 0$

Now, $A^2 - 6A + 17I_2 = 0 \quad \Rightarrow \quad A^2 - 6A = -17I_2$

Multiplying both sides by A^{-1}, we have

$$A - 6I_2 = -17A^{-1}$$

$$\therefore \quad A^{-1} = \frac{1}{17}(6I_2 - A) = \frac{1}{17}\left\{\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\right\} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}.$$

Q.7. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **and** I **is the identity matrix of order 2, then show that** $A^2 = 4A - 3I$**. Hence find** A^{-1}**.**

Ans.

Here, $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad A^2 = A.A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad ...(i)$$

$$\text{Also,} \quad 4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we get $A^2 = 4A - 3I$

Now, we have $A^2 = 4A - 3I$

Pre-multiplying both sides by A^{-1}

$$A^{-1} . A^2 = A^{-1} . (4A - 3I)$$

$$\Rightarrow \quad (A^{-1} . A) . A = 4 A^{-1} . A - 3 A^{-1} . I$$

$$\Rightarrow \quad IA = 4I - 3A^{-1}$$

$$\Rightarrow \quad A = 4I - 3A^{-1}$$

$$\Rightarrow \quad 3A^{-1} = 4I - A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{3}\left(4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) \Rightarrow$$

$$\frac{1}{3}\begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

Q.8. Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$**, express A as a sum of two matrices such that one is symmetric and other is skew symmetric.**

Ans.

A can be expressed as

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A'), \qquad ...(i) \qquad \left[\because \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(A + A' + A - A') = \frac{1}{2} \times 2A = A\right]$$

Where $A + A'$ and $A - A'$ are symmetric and skew symmetric matrices respectively.

$$\text{Now, } A + A' = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

Putting the values of $(A + A')$ and $(A - A')$ in (i), we get

$$A = \frac{1}{2}\begin{bmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 1 & 9/2 \\ 5/2 & 9/2 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 5/2 \\ 1 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix}$$

Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. If $A = \begin{bmatrix} 3 & -4 \\ 7 & 8 \end{bmatrix}$**, show that $A - A^T$ is a skew symmetric matrix where A^T is the transpose of matrix A.**

Ans.

Given: $A = \begin{bmatrix} 3 & -4 \\ 7 & 8 \end{bmatrix} \quad \therefore \quad A^T = \begin{bmatrix} 3 & 7 \\ -4 & 8 \end{bmatrix}$

$$A - A^T = \begin{bmatrix} 3 & -4 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & -11 \\ 11 & 0 \end{bmatrix}$$

Also, $(A - A^T)^T = \begin{bmatrix} 0 & -11 \\ 11 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 11 \\ -11 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -11 \\ 11 & 0 \end{bmatrix} = -(A - A^T)$

$\Rightarrow \quad (A - A^T)^T$ is a skew symmetric matrix.

Q.2. Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using this equation, find A^{-1}.

Ans.

We have, $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$$4A = 4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}$$

Hence, $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now, $A^2 - 4A + I = O$

Therefore, $A.A - 4A = -I$

or $A.A(A^{-1}) - 4AA^{-1} = -IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A\,(A\,A^{-1}) - 4I = -A^{-1}$

or $A - 4I = -A^{-1}$ $[A.A^{-1} = I \text{ and } IA = AI = A]$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Q.3. Solve the following:

Q. Prove that the sum of two skew-symmetric matrices is a skew-symmetric matrix.

Ans. Let A and B be two skew-symmetric matrices.

Then, $A' = -A$ and $B' = -B$.

$\therefore \quad (A + B)' = (A' + B') = (-A) + (-B) = -(A + B)$

Hence, $(A + B)$ is again a skew-symmetric.

Q. Express the following matrix as the sum of a symmetric and a skew-symmetric matrix.

$$\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$$

Ans.

Let $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ and $A' = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$

Let $P = \frac{A+A'}{2} = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}$ and $P' = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} = P,$

Hence, $\frac{A+A'}{2}$ is a symmetric matrix.

Now, $Q = \frac{A-A'}{2} = \frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$

Also, $Q' = \frac{1}{2}\begin{bmatrix} 0 & -9 & -9 \\ 9 & 0 & 3 \\ 9 & -3 & 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix} = -Q,$

Hence, $\frac{A-A'}{2}$ is a skew-symmetric matrix.

$$\therefore \quad P + Q = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ -12 & 16 & 6 \\ -8 & 12 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} = A$$

Hence, $A = \left(\frac{A+A'}{2}\right) + \left(\frac{A-A'}{2}\right)$

= Symmetric matrix + Skew-symmetric matrix.

Q.4. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ **Then show that $A^2 - 4A + 71 = 0$. Using this result calculate A^5.**

Ans.

Here, $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$$

Now, $A^2 - 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \text{ (zero matrix)}$$

$\Rightarrow \quad A^2 - 4A + 7I = 0 \quad \Rightarrow \quad A^2 = 4A - 7I$

$\Rightarrow \quad A.A^2 = 4A.A - 7A.I$ [Pre multiplying by A]

$\Rightarrow \quad A^3 = 4A^2 - 7A$ [$AI = A$]

$\Rightarrow \quad A^3 = 4(4A - 7I) - 7A$ [Putting the value of A^2]

$\Rightarrow \quad A^3 = 16A - 28I - 7A$

$\Rightarrow \quad A^3 = 9A - 28I$

$\Rightarrow \quad A.A^3 = 9A.A - 28A.I$ [Pre multiplying by A]

$\Rightarrow \quad A^4 = 9A^2 - 28A$

$\Rightarrow \quad A^4 = 9(4A - 7I) - 28A$ [Putting the value of A^2]

$\Rightarrow \quad A^4 = 8A - 63I$

$\Rightarrow \quad A.A^4 = 8A^2 - 63A$ [Pre multiplying by A]

$\Rightarrow \quad A^5 = 8(4A - 7I) - 63A = -31A - 56I$

$$= -31 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$

Q.5. If $A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in N$.

Ans.

We shall prove the result by using the principle of mathematical induction

When $n = 1$, we have

$$A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Thus, the result is true for $n = 1$.

Let the result be true for $n = m$.

Then $A^m = \begin{bmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{bmatrix}$

$$\therefore \quad A^{m+1} = A.A^m = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos m\theta - \sin\theta.\sin m\theta & \cos\theta\sin m\theta + \sin\theta\cos m\theta \\ -\sin\theta\cos m\theta - \cos\theta.\sin m\theta & -\sin\theta\sin m\theta + \cos\theta\cos m\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + m\theta) & \sin(\theta + m\theta) \\ -\sin(\theta + m\theta) & \cos(\theta + m\theta) \end{bmatrix} = \begin{bmatrix} \cos(m+1)\theta & \sin(m+1)\theta \\ -\sin(m+1)\theta & \cos(m+1)\theta \end{bmatrix}$$

Thus, the result is true for $n = (m + 1)$, whenever it is true for $n = m$.

Hence, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ for all $n \in N$.

Q.6. Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and skew-symmetric matrix.

Ans.

Let A be any square matrix. Then,

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q, \quad \text{(say)},$$

where, $P = \frac{1}{2}(A + A^T), Q = \frac{1}{2}(A - A^T)$.

Now, $P^T = \left(\frac{1}{2}(A + A^T)\right)^T$ $\quad [\because (KT)^T = K.A^T]$

$\Rightarrow \quad P^T = \frac{1}{2}[A^T + (A^T)^T]$ $\quad [\because (A + B)^T = A^T + B^T]$

$\Rightarrow \quad P^T = \frac{1}{2}(A^T + A)$ $\quad [\because (A^T)^T = A]$

$\Rightarrow \quad P^T = \frac{1}{2}(A + A^T) = P$

$\therefore$ P is symmetric matrix.

Also, $Q^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}[A^T - A]$

$\Rightarrow \quad Q^T = -\frac{1}{2}[A - A^T] = -Q$

$\therefore$ Q is skew-symmetric matrix.

Thus, $A = P + Q$, where P is a symmetric matrix and Q is a skew-symmetric matrix.

Hence, A is expressible as the sum of a symmetric and a skew-symmetric matrix.

Uniqueness: If possible, let $A = R + S$, where R is symmetric and S is skew-symmetric, then,

$$A^T = (R + S)^T = R^T + S^T$$

$\Rightarrow \quad A^T = R - S$ $\quad [\because R^T = R \text{ and } S^T = -S]$

Now, $A = R + S$ and $A^T = R - S$

$\Rightarrow \quad R = \frac{1}{2}[A + A^T] = P, \; S = \frac{1}{2}(A - A^T) = Q$

Hence, A is uniquely expressible as the sum of a symmetric and a skew-symmetric matrix.