Diffraction of Light (Part - 1)

Q.97. A plane light wave falls normally on a diaphragm with round aperture opening the first N Fresnel zones for a point P on a screen located at a distance b from the diaphragm. The wavelength of light is equal to 2n,. Find the intensity of light /0 in front of the diaphragm if the distribution of intensity of light I (r) on the screen is known. Here r is the distance from the point P.

Ans.

The radius of the periphery of the N^{th} Fresnel zone is

$$r_N = \sqrt{N b \lambda}$$

Then by conservation of energy

$$I_0 \pi (\sqrt{N b \lambda})^2 = \int_0^{\infty} 2 \pi r \, dr \, I(r)$$

Here r is the distance from the point P.

Thus
$$I_0 = \frac{2}{N b \lambda} \int_0^{\infty} r \, dr \, I(r).$$

Q.98. A point source of light with wavelength λ = 0.50 p,m is located at a distance a = 100 cm in front of a diaphragm with round aperture of radius r = 1.0 mm. Find the distance b between the diaphragm and the observation point for which the number of Fresnel zones in the aperture equals k = 3.

Ans. By definition

$$r_k^2 = k \frac{a b \lambda}{a + b}$$

For the periphery of the k^{th} zone. Then

$$a r_k^2 + b r_k^2 = k a b \lambda$$

So
$$b = \frac{a r_k^2}{k a \lambda - r_k^2} = \frac{a r^2}{k a \lambda - r^2} = 2 \text{ metre}.$$

on putting the values. (It is given that $r = r_k$) for $k = 3$).

Q.99. A diaphragm with round aperture, whose radius r can be varied during the experiment, is placed between a point source of light and a screen. The distances from the diaphragm to the source and the screen are equal to a = 100 cm and b = 125 cm. Determine the wavelength of light if the intensity maximum at the centre

of the diffraction pattern of the screen is observed at r_1 = 1.00 mm and the next maximum at r_2 = 1.29 mm

Ans. Suppose maximum intensity is obtained when the aperture contains k zones. Then a minimum will be obtained for k + 1 zones. Another maximum will be obtained for k + 2 zones. Hence

$$r_1^2 = k\lambda \frac{ab}{a+b}$$

$$r_2^2 = (k+2)\lambda \frac{ab}{a+b}$$

Thus $\lambda = \frac{a+b}{2ab}(r_2^2 - r_1^2) = 0{\cdot}598\ \mu m$

On putting the values.

Q.100. A plane monochromatic light wave with intensity I_0 falls normally on an opaque screen with a round aperture. What is the intensity of light I behind the screen at the point for which the aperture (a) is equal to the first Fresnel zone; to the internal half of the first zone; (b) was made equal to the first Fresnel zone and then half of it was closed (along the diameter)?

Ans. (a) When the aperture is equal to the first Fresnel Zone
The amplitude is Ax and should be compared with the amplitude y when the aperture is very wide. If I0 is the intensity in the second case the intensity in the first case will be $4I_o$

When the aperture is equal to the internal half of the first zone Suppose A_{in} and A_{out} are the amplitudes due to the two halves o f the first Fresnel zone. A_{in} and A_{out} differ in

phase by $\frac{\pi}{2}$ because only half the Fresnel zone in involved. Also in magnitude $|A_{in}| = |A_{out}|$ then

$$A_1^2 = 2|A_{in}|^2 \quad \text{so} \quad |A_{in}|^2 = \frac{A_1^2}{2}$$

Hence following the aigument of the first case. $I_{in} = 2I_0$

(b) The aperture was made equal to the first Fresnel zone and then half of it was closed

along a diameter. In this case the amplitude of vibration is $\frac{A_1}{2}$. Thus
$I = I_0$.

Q.101. A plane monochromatic light wave with intensity Iofalls normally on an opaque disc closing the first Fresnel zone for the observation point P. What did the intensity of light I at the point P become equal to after (a) half of the disc (along the diameter) was removed; (b) half of the external half of the first Fresnel zone was removed (along the diameter)?

Ans. (a) Suppose the disc does not obstruct light at all. Then

$$A_{disc} + A_{remainder} = \frac{1}{2} A_{disc}$$

(because the disc covers the first Fresnel zone only).

$$\text{So} A_{remainder} = -\frac{1}{2} A_{disc}$$

Hence the amplitude when half of the disc is removed along a diameter

$$= \frac{1}{2} A_{disc} + A_{remainder} = \frac{1}{2} A_{disc} - \frac{1}{2} A_{disc} = 0$$

Hence $I = 0$.

(b) In this case

$$A = \frac{1}{2} A_{external} + A_{remainder}$$
$$= \frac{1}{2} A_{external} - \frac{1}{2} A_{disc}$$

We write

$$A_{disc} = A_{in} + i A_{out}$$

where A_{in} (A_{out}) stands for $A_{internal}$ ($A_{external}$). The factor i takes account of the $\frac{\pi}{2}$ phase difference between two halves of the first Fresnel zone. Thus

$$A = -\frac{1}{2} A_{in} \quad \text{and} \quad I = \frac{1}{4} A_{in}^2$$

On the other hand

$$I_0 = \frac{1}{4}(A_{in}^2 + A_{out}^2) = \frac{1}{2} A_{in}^2$$

So $I = \frac{1}{2} I_0.$

Q.102. A plane monochromatic light wave with intensity /0 falls normally on the surfaces of the opaque screens shown in Fig. 5.20. Find the intensity of light I at a point P

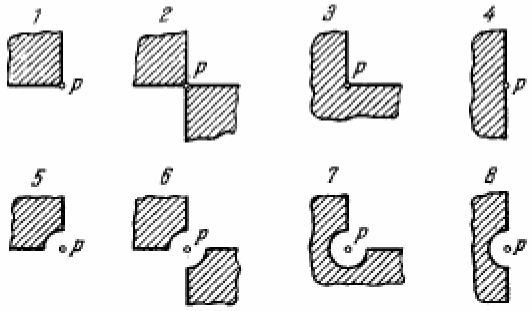


Fig. 5.20.

(a) located behind the corner points of screens 1-3 and behind the edge of half-plane 4;
(b) for which the rounded-off edge of screens 5-8 coincides with the boundary of the first Fresnel zone. Derive the general formula describing the results obtained for screens 1-4; the same, for screens 5-8.

Ans. When the screen is fully transparent, the. Amplitude of vibrations is $\frac{1}{2}A_1$ (with intensity

$I_0 = \frac{1}{4}A_1^2$).

(a) (1) In this case $A = \frac{3}{4}\left(\frac{1}{2}A_1\right)$ so squaring $I = \frac{9}{16}I_0$

(2) In this case $\frac{1}{2}$ of the plane is blacked out so

$$A = \frac{1}{2}\left(\frac{1}{2}A_1\right) \quad \text{and} \quad I = \frac{1}{4}I_0$$

(3) In this case $A = \frac{1}{4}(A_1/2)$ and $I = \frac{1}{16}I_0$.

(4) In this case $A = \frac{1}{2}\left(\frac{1}{2}A_1\right)$ again and $I = \frac{1}{4}I_0$ so $I_4 = \frac{I}{2}$

In general we get $I(\varphi) = I_0\left(1 - \left(\frac{\varphi}{2\pi}\right)\right)^2$

where φ is the total angle blocked out by the screen.

(b) (5) Here $A = \frac{3}{4}\left(\frac{1}{2}A_1\right) + \frac{1}{4}A_1$

A_1 being the contribution of the first Fresnel zone.

Thus $$A = \frac{5}{8}A_1 \quad \text{and} \quad I = \frac{25}{16}I_0$$

(6) $A = \frac{1}{2}\left(\frac{1}{2}A_1\right) + \frac{1}{2}A_1 = \frac{3}{4}A_1$ and $I = \frac{9}{4}I_0$

$$(7)\ A = \frac{1}{4}\left(\frac{1}{2}A_1\right) + \frac{3}{4}A_1 = \frac{7}{8}A_1 \text{ and } I = \frac{49}{16}I_0$$

$$(8)\ A = \frac{1}{2}\left(\frac{1}{2}A_1\right) + \frac{1}{2}A_1 = \frac{3}{4}A_1 \text{ and } I = \frac{9}{4}I_0\ (I_8 = I_6)$$

In 5 to 8 the first term in the expression for the amplitude is the contribution of the plane part and the second term gives the expression for the Fresnel zone part. In general in (5) to

$$(8)\ I = I_0\left(1 + \left(\frac{\varphi}{2\pi}\right)\right)^2 \text{ when } \varphi$$ is the angle covered by the screen.

Q.103. A plane light wave with wavelength λ = 0.60 1.t.m falls normally on a sufficiently large glass plate having a round recess on the opposite side (Fig. 5.21). For the observation point P that recess corresponds to the first one and a half Fresnel zones. Find the depth h of the recess at which the intensity of light at the point P is (a) maximum; (b) minimum; (c) equal to the intensity of incident light.

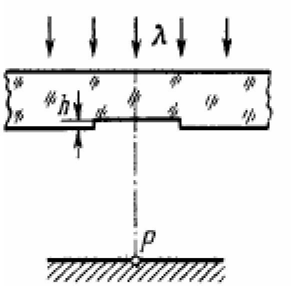
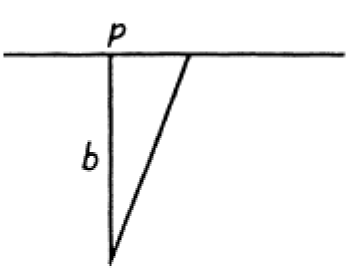


Fig. 5.21.

Ans. We would require the contribution to the amplitude of a wave at a point from half a Fresnel zone. For this we proceed directly from the Fresnel Huygens principle. The complex amplitude is written as

$$E = \int K(\varphi)\,\frac{a_0}{r}\,e^{-ikr}\,dS$$



Here $K(\varphi)$ is a factor which depends on the angle φ between a normal $\vec{n}$ to the area dS and the direction from dS to the point P and r is the distance from the element dS to P.

We see that for the first Fresnal zone

$$\left(\text{using } r = b + \frac{\rho^2}{2b} \left(\text{for } \sqrt{\rho^2 + b^2}\right)\right)$$

$$E = \frac{a_0}{b} \int_0^{\sqrt{b\lambda}} e^{-ikb - ik\rho^2/2b} \, 2\pi\rho \, d\rho \quad (K(\varphi) = 1)$$

For the first Fresnel zone $r = b + \lambda/2$ so $r^2 = b^2 + b\lambda$ and $\rho^2 = b\lambda$.

Thus

$$\begin{aligned} E &= \frac{a_0}{b} e^{-ikb} \, 2\pi \int_0^{\frac{b\lambda}{2}} e^{-i\frac{kx}{b}} \, dx \\ &= \frac{a_0}{b} 2\pi e^{-ikb} \frac{e^{-ik\lambda/2} - 1}{-ik/b} \\ &= \frac{a_0}{k} 2\pi i e^{-ikb} (-2) = -\frac{4\pi}{k} i a_0 e^{-ikb} = A_1 \end{aligned}$$

For the next half zone

$$\begin{aligned} E &= \frac{a_0}{b} e^{-ikb} \, 2\pi \int_{\frac{b\lambda}{2}}^{\frac{3b\lambda}{4}} e^{-ikx/b} \, dx x \\ &= \frac{a_0}{k} 2\pi i e^{-ikb} \left(e^{-i\frac{3k\lambda}{4}} - e^{-ik\lambda/2}\right) \\ &= \frac{a_0}{k} 2\pi i \, e^{-ikb} \, (+1+i) = -\frac{A_1(1+i)}{2} \end{aligned}$$

If we calculate the contribution of the full 2nd Fresnel zone we will get - Av If we take account of the factors K (φ) and $\frac{1}{r}$ which decrease monotonically we expect the contribution to change to - A_2. Thus we write for the contribution of the half zones in the 2nd Fresnel zone as

$$-\frac{A_2(1+i)}{2} \text{ and } -\frac{A_2(1-i)}{2}$$

The part lying in the recess has an extra phase difference equal to $-\delta = -\frac{2\pi}{\lambda}(n-1)h.$ Thus the full amplitude is (note that the correct form is e^{-ikr})

$$\left(A_1 - \frac{A_2}{2}(1+i)\right)e^{+i\delta} - \frac{A_2}{2}(1-i) + A_3 - A_4 + \dots$$
$$= \left(\frac{A_1}{2}(1-i)\right)e^{+i\delta} - \frac{A_2}{2}(1-i) + \frac{A_3}{2}$$

$$= \left(\frac{A_1}{2}(1-i)\right)e^{+i\delta} + i\frac{A_1}{2} \text{ (as } A_2 = A_3 = A_1\text{) and } A_3 - A_4 + A_5 \dots = \frac{A_3}{2}.$$

The corresponding intensity is

$$I = \frac{A_1^2}{4}\left[(1-i)e^{+i\delta} + \frac{i}{e}\right]\left[(1+i)e^{-i\delta} - i\right]$$
$$= I_0[3 - 2\cos\delta + 2\sin\delta] = I_0\left[3 + 2\sqrt{2}\sin\left(\delta - \frac{\pi}{4}\right)\right]$$

(a) For maximum intensity sin

$$\sin\left(\delta - \frac{\pi}{4}\right) = +1$$

or $$\delta - \frac{\pi}{4} = 2k\pi + \frac{\pi}{2}, \quad k = 0, 1, 2, \dots$$

$$\delta = 2k\pi + \frac{3\pi}{4} = \frac{2\pi}{\lambda}(n-1)h$$

so $$h = \frac{\lambda}{n-1}\left(k + \frac{3}{8}\right)$$

(b) For minimum intensity

$$\sin\left(\delta - \frac{\pi}{4}\right) = 1$$

$$\delta - \frac{\pi}{4} = 2k\pi + \frac{3\pi}{2} \quad \text{or} \quad \delta = 2k\pi + \frac{7\pi}{4}$$

$$h = \frac{\lambda}{n-1}\left(k + \frac{7\pi}{8}\right)$$

(c) For $I = I_0$, $\left.\begin{matrix}\cos\delta = 0 \\ \sin\delta = -1\end{matrix}\right\}$ or $\left\{\begin{matrix}\sin\delta = 0 \\ \cos\delta = +1\end{matrix}\right.$

Thus $\delta = 2k\pi \quad h = \dfrac{k\lambda}{n-1}$

or $\delta = 2k\pi + \dfrac{3\pi}{2}, \; h = \dfrac{\lambda}{n-1}\left(k + \dfrac{3\pi}{4}\right)$

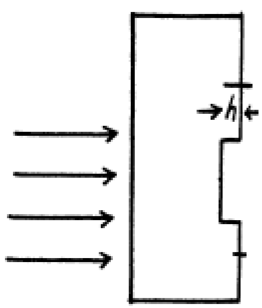
Q.104. A plane light wave with wavelength λ and intensity I_0 falls normally on a large glass plate whose opposite side serves as an opaque screen with a round aperture equal to the first Fresnel zone for the observation point P. In the middle of the aperture there is a round recess equal to half the Fresnel zone. What must the depth h of that recess be for the intensity of light at the point P to be the highest? What is this intensity equal to?

Ans. The contribution to the wave amplitude of the inner half-zone is

$$\frac{2\pi a_0 e^{-ikb}}{b}\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\rho\, d\rho$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}\int_0^{b\lambda/4} e^{-ikx/b}dx$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}\left(e^{-ik\lambda/4} - 1\right) \times \frac{1}{\frac{-ik}{b}}$$



$$= \frac{2\pi i a_0 e^{-ikb}}{k}(-i-1) = +\frac{A_1}{2}(1+i)$$

With phase factor this becomes $\frac{A_1}{2}(1+i)e^{i\delta}$ where $\delta = \frac{2\pi}{\lambda}(n-1)h.$ The contribution of the remaining aperture is $\frac{A_1}{2}(1-i)$

(so that the sum of the two parts when $\delta = 0$ is A_1)

Thus the complete amplitude is

$$\frac{A_1}{2}(1+i)e^{i\delta} + \frac{A_1}{2}(1-i)$$

And the intensity is

$$I = I_0[(1+i)e^{i\delta} + (1-i)][(1-i)e^{-i\delta} + (1+i)]$$

$$= I_0[2 + 2 + (1-i)^2 e^{-i\delta} + (1+i)^2 e^{i\delta}]$$

$$= I_0[4 - 2ie^{-i\delta} + 2ie^{i\delta}] = I_0(4 - 4\sin\delta)$$

Here $I_0 = \frac{A_1^2}{4}$ .s the intensity of the incident light which is the same as the intensity due to an aperture of infinite extent (and no recess). Now

l is maximum when $\sin\delta = -1$

$$\delta = 2k\pi + \frac{3\pi}{2}$$

$$h = \frac{\lambda}{n-1}\left(k + \frac{3}{4}\right) \quad \text{and (b)} \quad I_{max} = 8I_0$$

Q.105. A plane light wave with wavelength λ = 0.57 um falls normally on a surface of a glass (n = 1.60) disc which shuts one and a half Fresnel zones for the observation point P. What must the minimum thickness of that disc be for the intensity of light at the point P to be the highest? Take into account the interference of light on its passing through the disc.

Ans. We follow the argument of 5.103. We find that the contribution of the first Fresnel zone is

$$A_1 = -\frac{4\pi i}{k} a_0 e^{-ikb}$$

For the next h a lf zone it i $-\frac{A_2}{2}(1+i)$

(The contribution of the remaining part of the 2nd Fresnel zone will be $-\frac{A_2}{2}(1-i))$

If the disc has a thickness h, the extra phase difference suffered by the light wave in passing through' the disc will be

$$\delta = \frac{2\pi}{\lambda}(n-1)h.$$

Thus the amplitude at P will be

$$E_P = \left(A_1 - \frac{A_2}{2}(1+i)\right)e^{-i\delta} - \frac{A_2}{2}(1-i) + A_3 - A_4 - A_5 +$$

$$= \left(\frac{A_1(1-i)}{2}\right)e^{-i\delta} + \frac{iA_1}{2} = \frac{A_1}{2}[(1-i)e^{-i\delta} + i]$$

The corresponding intensity will be

$$I = I_0(3 - 2\cos\delta - 2\sin\delta) = I_0\left(3 - 2\sqrt{2}\sin\left(\delta + \frac{\pi}{4}\right)\right)$$

The intensity will be a maximum when

$$\sin\left(\delta + \frac{\pi}{4}\right) = -1$$

or
$$\delta + \frac{\pi}{4} = 2k\pi + \frac{3\pi}{2}$$

i.e
$$\delta = \left(k + \frac{5}{8}\right).2\pi$$

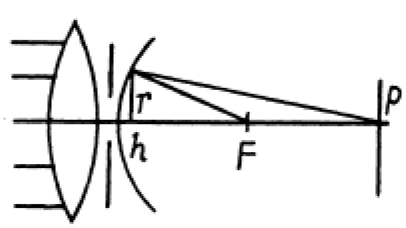
so
$$h = \frac{\lambda}{n-1}\left(k + \frac{5}{8}\right), \ k = 0, 1, 2, \ldots$$

Note :- It is not clear why $k = 2$ for h_{min}. The normal choice will be $k = 0$. If we take $k = 0$ we get $h_{min} = 0{\cdot}59\ \mu m$.

Q.106. A plane light wave with wavelength λ = 0.54 um goes through a thin converging lens with focal length f = 50 cm and an aperture stop fixed immediately after the lens, and reaches a screen placed at a distance b = 75 cm from the aperture stop. At what aperture radii has the centre of the diffraction pattern on the screen the maximum illuminance?

Ans. Here the focal point acts as a virtual source of light. This means that we can take spherical waves converging towards F. Let us divide these waves into Fresnel zones just after they emerge from the stop. We write

$$r^2 = f^2 - (f-h)^2 = (b - m\lambda/2)^2 - (b-h)^2$$



Here r is the radius of the m^{th} fresnel zone and A is the distance to the left of the foot of the perpendicular. Thus

$$r^2 = 2fh = -bm\lambda + 2bh$$

So $$h = bm\lambda/2(b-f)$$

and $$r^2 = fbm\lambda/(b-f).$$

The intensity maxima are observed when an odd number of Fresnel zones are exposed by the stop. Thus

$$r_k = \sqrt{\frac{kbf\lambda}{b-f}} \quad \text{where} \quad k = 1, 3, 5, \ldots$$

Q.107. A plane monochromatic light wave falls normally on a round aperture. At a distance b = 9.0 m from it there is a screen showing a certain diffraction pattern. The aperture diameter was decreased η = 3.0 times. Find the new distance b' at which the screen should be positioned to obtain the diffraction pattern similar to the previous one but diminished η times.

Ans. or the radius of the periphery of the k^{th} zone we have

$$r_k = \sqrt{k\lambda\frac{ab}{a+b}} = \sqrt{k\lambda b} \quad \text{if } a = \infty.$$

If the aperture diameter is reduced η times it will produce a similar detraction pattern (reduced η times) if the radii of the Fresnel zones are also η times less. Thus

$$r'_k = r_k/\eta$$

This requires $b' = b/\eta^2$.

Q.108. An opaque ball of diameter D = 40 mm is placed between a source of light with wavelength λ = 0.55 um and a photographic plate. The distance between the source and the ball is equal to a = 12 m and that between the ball and the photographic plate is equal to b = 18 m. Find:
(a) the image dimension y' on the plate if the transverse dimension of the source is y = 6.0 mm;
(b) the minimum height of irregularities, covering the surface of the ball at random, at which the ball obstructs light.
Note. As calculations and experience show, that happens when the height of irregularities is comparable with the width of the Fresnel zone along which the edge of an opaque screen passes

Ans. (a) If a point source is placed before an opaque ball, the diffraction pattern consists of a bright spot inside a dark disc followed by fringes. The bright spot is on the line joining the point source and the centre of the ball. When the object is a finite source of transverse dimension, every point of the source has its corresponding image on the line joining that point and the centre of the ball. Thus the transverse dimension of the image is given by

$$y' = \frac{b}{a}y = 9 \text{ mm.}$$

(b) The minimum height of the irregularities covering the surface of the ball at random, at which the ball obstructs light is according to the note at the end of the problem, corn* parable with the width of the Fresnel zone along which the edge of opaque screen passes. SO

$$h_{min} \approx \Delta r$$

To find Δr we note that

$$r^2 = \frac{k\lambda a b}{a+b}$$

$$2r\Delta r = D\Delta r = \frac{\lambda a b}{a+b}\Delta k$$

Where D = diameter of the disc (= diameter of the last Fresnel zone) and Δk = 1 Thus

$$\text{Thus } h_{min} = \frac{\lambda a b}{D(a+b)} = 0{\cdot}099 \text{ mm.}$$

Q.109. A point source of monochromatic light is positioned in front of a zone plate at a distance a = 1.5 m from it. The image of the source is formed at a distance b = 1.0 m from the plate. Find the focal length of the zone plate.

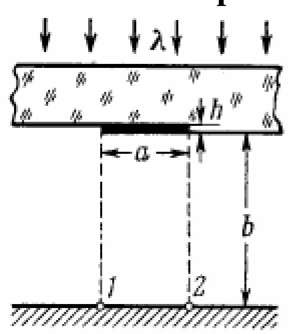


Fig. 5.23.

Ans.

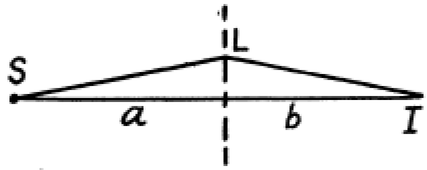
1st, 2nd, 3rd..... Fresnel zones.

Let r_1 = radius of the central undarkened circle. Then for this to be first Fresnel zone in the present case, we must have

$$SL + LI - SI = \lambda/2$$

Thus if r_1 is the radius of the periphery of the first zone

$$\sqrt{a^2+r_1^2}+\sqrt{b^2+r_1^2}-(a+b)=\frac{\lambda}{2}$$



or $$\frac{r_1^2}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{\lambda}{2} \quad \text{or} \quad \frac{1}{a}+\frac{1}{b}=\frac{1}{r_1^2/\lambda}$$

It is clear that the plate is acting as a lens of focal length

$$f_1=\frac{r_1^2}{\lambda}=\frac{ab}{a+b}=\cdot 6 \text{ metre.}$$

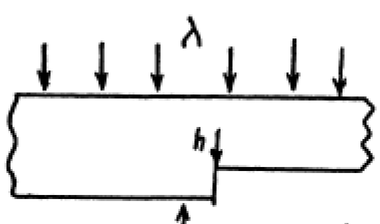
This is the principle focal length. Other maxima are obtained when

$$SL+LI-SI=3\frac{\lambda}{2},\ 5\frac{\lambda}{2},\ldots$$

These focal lengths are also $\frac{r_1^2}{3\lambda}, \frac{r_1^2}{5\lambda}, \ldots$

Q.110. A plane light wave with wave- length λ = 0.60 um and intensity I, falls normally on a large glass plate whose side view is shown in Fig. 5.22. At what height h of the ledge will the intensity of light at points located directly below be
(a) minimum;
(b) twice as low as I_0 (the losses due to reflection are to be neglected).

Ans. Just below the edge the amplitude of the wave is given by



$$A=\frac{1}{2}(A_1-A_2+A_3-A_4+\ldots)e^{-i\delta}$$
$$+\frac{1}{2}(A_1-A_2+A_3-A_4+\ldots)$$

Here the quantity in the brackets is the contribution of various Fresnel zones; the factor $\frac{1}{2}$ is to take account of the division of the plate into two parts by the ledge; the phase factor 5 is given by

$$\delta = \frac{2\pi}{\lambda} h (n-1)$$

And takes into account the extra length traversed by the waves on the left.

Using $A_1 - A_2 + A_3 - A_4 + \dots = \frac{A_1}{2}$

we get $A = \frac{A_1}{4}(1 + e^{i\delta})$

And the corresponding intensity is

$$I = I_0 \frac{1 + \cos\delta}{2}, \quad \text{where } I_0 \propto \left(\frac{A_1}{2}\right)^2$$

So

$$\cos\delta = -1$$

$$\delta = (2k+1)\pi$$

and

$$h = (2k+1)\frac{\lambda}{2(n-1)}, \quad k = 0, 1, 2, \dots$$

using $n = 1{\cdot}5$, $\lambda = 0{\cdot}60\,\mu m$

$$h = 0{\cdot}60\,(2k+1)\,\mu\text{m}.$$

(b) $I = I_0/2$ when $\cos\delta = 0$

or

$$\delta = k\pi + \frac{\pi}{2} = (2k+1)\frac{\pi}{2}$$

Thus in this case $h = 0{\cdot}30\,(2k+1)\,\mu\text{m}.$

Q.111. A plane monochromatic light wave falls normally on an opaque half-plane. A screen is located at a distance b = 100 cm behind the half-plane. Making use of the Cornu spiral (Fig. 5.19), find: (a) the ratio of intensities of the first maximum and the neighbouring minimum; (b) the wavelength of light if the first two maxima are separated by a distance Δx = 0.63 mm.

Ans. (a) From the Cornu's spiral, the intensity of t^e first maximum is given as
$I_{max,1} = 1{\cdot}37\, I_0$

And the intensity of the first minimum is given by
$I_{min} = 0{\cdot}78\, I_0$
so the required ratio is

$$\frac{I_{max}}{I_{min}} = 1.76$$

(b) The value of the distance x is related to the parameter v in Fresnel's integral by

$$v = x\sqrt{\frac{2}{b\lambda}}$$

For the first two maxima the distances x_1 , x_2 are related to the parameters v_1, v_2 by

$$x_1 = \sqrt{\frac{b\lambda}{2}}\,v_1,\ x_2 = \sqrt{\frac{b\lambda}{2}}\,v_2$$

Thus $$(v_2 - v_1)\sqrt{\frac{b\lambda}{2}} = x_2 - x_1 = \Delta x$$

or $$\lambda = \frac{2}{b}\left(\frac{\Delta x}{v_2 - v_1}\right)^2$$

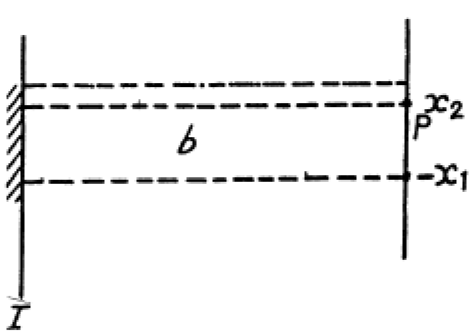
From the Cornu's spiral the positions of the maxima are

$$v_1 = 1.22,\ v_2 = 2.34,\ v_3 = 3.08 \text{ etc}$$

Thus $$\lambda = \frac{2}{b}\left(\frac{\Delta x}{1.12}\right)^2 = 0.63\ \mu\text{m}.$$

Q.112. A plane light wave with wavelength 0.60 pm falls normally on a long opaque strip 0.70 mm wide. Behind it a screen is placed at a distance 100 cm. Using Fig. 5.19, find the ratio of intensities of light in the middle of the diffraction pattern and at the edge of the geometrical shadow.

Ans. We shall use the equation written down in 5.103, the Fresnel-Huygens formula.



Suppose we want to find the intensity at P which is such that the coordinates of the edges (x-coordinates) with respect to P are x_2 and - x_1. Then, the amplitude at P is

$$E = \int K(\varphi)\, \frac{a_0}{r}\, e^{-ikr}\, dS \qquad (1)$$

We write $dS = dx\,dy$, y is to integrated from $-\infty + 0 \to +\infty$. We write

$$r = b + \frac{x^2 + y^2}{2b} \qquad (1)$$

(r is the distance of the element of surface on I from P. It is $\sqrt{b^2 + x^2 + y^2}$ and hence approximately (1)). We then get

$$E = A_0(b)\left[\int_{x_2}^{\infty} e^{-ikx^2/2b}\, dx + \int_{-\infty}^{-x_1} e^{-ikx^2/2b}\, dx\right]$$

$$= A'_0(b)\left[\int_{v_2}^{\infty} e^{-i\frac{\pi u^2}{2}}\, du + \int_{-\infty}^{-v_1} e^{-i\pi u^2/2}\, du\right]$$

where $\quad v_2 = \sqrt{\frac{2}{b\lambda}}\, x_2,\ v_1 = \sqrt{\frac{2}{b\lambda}}\, x_1$

The intensity is the square of the amplitude. In our case, at the centre

$$v_1 = v_2 = \sqrt{\frac{2}{b\lambda}} \cdot \frac{a}{2} = \sqrt{\frac{a^2}{2b\lambda}} = 0{\cdot}64$$

(a = width of the strip = 0·7 mm, b = 100 cm, λ = 0·60 μm)

At, say, the lower edge $\quad v_1 = 0,\ v_2 = 1{\cdot}28$

Thus

$$\frac{I_{\text{centre}}}{I_{\text{edge}}} = \left|\frac{\int_{0\cdot64}^{\infty} e^{-i\pi u^2/2}\, du + \int_{-\infty}^{-0\cdot64} e^{-i\pi u^2/2}\, du}{\int_{1\cdot28}^{\infty} e^{-i\pi u^2/2}\, du + \int_{-\infty}^{0} e^{-i\pi u^2/2}\, du}\right|^2 = 4\,\frac{\left(\frac{1}{2} - C(0{\cdot}64)\right)^2 + \left(\frac{1}{2} - S(0{\cdot}64)\right)^2}{(1 - C(1{\cdot}28))^2 + (1 - S(1{\cdot}28))^2}$$

Where

$$C(v) = \int_0^v \cos\frac{\pi u^2}{2}\, du$$

$$S(v) = \int_0^v \sin\frac{\pi u^2}{2}\, du$$

Rough evaluation of the integrals using cornu's spiral gives

$$\frac{I_{\text{centre}}}{I_{\text{edge}}} = 2{\cdot}4$$

(We have used $\int_0^\infty \cos\frac{\pi u^2}{2}\,du = \int_0^\infty \sin\frac{\pi u^2}{2}\,du = \frac{1}{2}$

$$C(0{\cdot}64) = 0{\cdot}62,\ S(0{\cdot}64) = 0{\cdot}15$$

$$C(1{\cdot}28) = 0{\cdot}65\ \ S(1{\cdot}28) = 0{\cdot}67$$

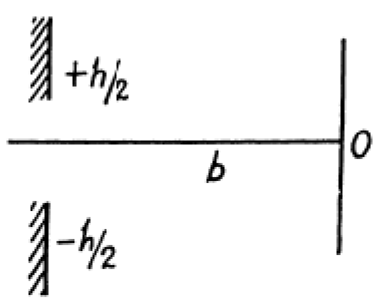
Q.113. A plane monochromatic light wave falls normally on a long rectangular slit behind which a screen is positioned at a distance b = 60 cm. First the width of the slit was adjusted so that in the middle of the diffraction pattern the lowest minimum was observed. After widening the slit by Δh = 0.70 mm, the next minimum was obtained in the centre of the pattern. Find the wavelength of light.

Ans. If the aperture has width A then the parameters (v , - v) associated with

associated with $\left(h/2,\ -\frac{h}{2}\right)$ are given by

$$v = \frac{h}{2}\sqrt{\frac{2}{b\lambda}} = h/\sqrt{2b\lambda}$$

The intensity of light at O on the screen is obtained as the square of the amplitude A of the wave at O which is



Thus $\qquad I = 2I_0\left((C(v))^2 + (S(v))^2\right)$

where $C(v)$ and $S(v)$ have been defined above and I_0 is the intensity at O due to an infinitety wide ($v = \infty$) aperture for then

$$I = 2I_0\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) = 2I_0 \times \frac{1}{2} = I_0$$

By definition v corresponds to the first minimum of the intensity. This means when we increase h to $h + \Delta h$, the corresponding $\nu_2 = \frac{h + \Delta h}{\sqrt{2 b \lambda}}$ relates to the second

minimum of intensity. From the cornu's spiral $\nu_2 = 2{\cdot}75$

Thus $\Delta h = \sqrt{2 b \lambda}\ (\nu_2 - \nu_1) = 0{\cdot}85 \sqrt{2 b \lambda}$

or $\lambda = \left(\frac{\Delta h}{0{\cdot}85}\right)^2 \frac{1}{2b} = \left(\frac{0{\cdot}70}{0{\cdot}85}\right)^2 \frac{1}{2 \times 0{\cdot}6} \mu\text{m} = 0{\cdot}565\ \mu\text{m}$

Q.114. A plane light wave with wavelength λ = 0.65 p.m falls normally on a large glass plate whose opposite side has a long rectangular recess 0.60 mm wide. Using Fig. 5.19, find the depth h of the recess at which the diffraction pattern on the screen 77 cm away from the plate has the maximum illuminance at its centre.

Ans. Let a = width of the recess and

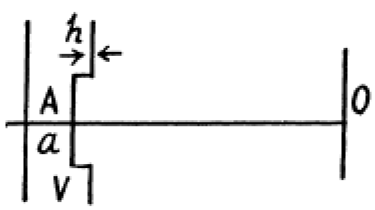
$$\nu = \frac{a}{2}\sqrt{\frac{2}{b\lambda}} = \frac{a}{\sqrt{2 b \lambda}} = \frac{0{\cdot}6}{\sqrt{2 \times 0{\cdot}77 \times 0{\cdot}65}} = 0{\cdot}60$$

be die parameter along Cornu's spiral corresponding to the half-width of the recess. The amplitude of the diffracted wave is given by

$$\sim \text{const}\left[e^{i\delta}\int_{-\nu}^{\nu} e^{-i\pi u^2/2}\,du + \int_{\nu}^{\infty} e^{-i\pi u^2/2}\,du + \int_{-\infty}^{\nu} e^{-i\pi u^2/2}\,du\right]$$

where $\delta = \frac{2\pi}{\lambda}(n-1)h$

Is the extra phase due to the recess. (Actually an extra phase $e^{-1\delta}$ appears outside the recess. When we take it out and absorb it in the constant we get the expression written). Thus the amplitude is



$$\sim \text{const}\left[(C(\nu) - iS(\nu))\,e^{i\delta} + \left(\frac{1}{2} - C(\nu)\right) - i\left(\frac{1}{2} - S(\nu)\right)\right]$$

From the Cornu's spiral, the coordinates corresponding to the parameter $\nu = 0{\cdot}60$ are

$$C(\nu) = 0{\cdot}57,\ S(\nu) = 0{\cdot}13$$

so the intensity at O is proportional to

$$\left|\left[(0{\cdot}57-0{\cdot}13\,i)\,e^{i\delta}-0{\cdot}07-i\,0{\cdot}37\right]\right|^2$$
$$=(0{\cdot}57^2+0{\cdot}13^2)+0{\cdot}07^2+0{\cdot}37^2$$
$$+(0{\cdot}57-0{\cdot}13\,i)(-0{\cdot}07+0{\cdot}37\,i)\,e^{i\delta}$$
$$+(0{\cdot}57+0{\cdot}13\,i)(-0{\cdot}07-i\,0{\cdot}37\,i)\,e^{-i\delta}$$

We write

$$0{\cdot}57\mp 0{\cdot}13\,i=0{\cdot}585\;e^{\mp i\alpha}\quad \alpha=12{\cdot}8^\circ$$
$$-0{\cdot}07\pm 0{\cdot}37\,i=0{\cdot}377\;e^{\pm i\beta}\quad \beta=100{\cdot}7^\circ$$

Thus the cross term is

$$2\times 0{\cdot}585\times 0{\cdot}377\ \cos(\delta+88^\circ)$$
$$\approx 2\times 0{\cdot}585\times 0{\cdot}377\ \cos\left(\delta+\frac{\pi}{2}\right)$$

For maximum intensity

$$\delta+\frac{\pi}{2}=2k'\pi,\quad k'=1,2,3,4,\ldots$$
$$=2(k+1)\pi,\ k=0,1,2,3,\ldots$$

Or $$\delta=2k\pi+\frac{3\pi}{2}$$

So $$h=\frac{\lambda}{n-1}\left(k+\frac{3}{4}\right)$$

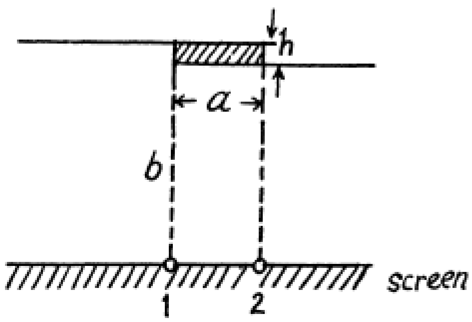
Q.115. A plane light wave with wavelength λ = 0.65 p.m falls normally on a large glass plate whose opposite side has a ledge and an opaque strip of width a = 0.30 mm (Fig. 5.23). A screen is placed at a distance b = 110 cm from the plate. The height h of the ledge is such that the intensity of light at point 2 of the

Fig. 5.24.

Fig. 5.25.

screen is the highest possible. Making use of Fig. 5.19, find the ratio of intensities at points 1 and 2.

Ans.



Using the method of problem 5.103 we can immediately write down the amplitudes at 1 and 2. We get:

At 1
$$\text{amplitude } A_1 \sim \text{const}\left[\int_{-\infty}^{0} e^{-i\pi u^2/2}\,du + e^{-i\delta}\int_{v}^{\infty} e^{-i\pi u^2/2}\,du\right]$$

At 2
$$\text{amplitude } A_2 \sim \text{const}\left[\int_{-\infty}^{-v} e^{-i\pi u^2/2}\,du + e^{-i\delta}\int_{0}^{\infty} e^{-i\pi u^2/2}\,du\right]$$

where $v = a\sqrt{\frac{2}{b\lambda}}$

is the parameter of Cornu's spiral and constant factor is common to 1 and 2. With the usual notation

$$C = C(v) = \int_0^v \cos\frac{\pi u^2}{2}\,du$$

$$S = S(v) = \int_0^v \sin\frac{\pi u^2}{2}\,du$$

And the result
$$\int_0^\infty \cos\frac{\pi u^2}{2}\,du = \int_0^\infty \sin\frac{\pi u^2}{2}\,du = \frac{1}{2}$$

We find the ratio of intensities as

$$\frac{I_2}{I_1} = \left|\frac{\left(\frac{1}{2}-C\right) - i\left(\frac{1}{2}-S\right) + e^{-i\delta}\frac{(1-i)}{2}}{\frac{1-i}{2} + e^{-i\delta}\left\{\left(\frac{1}{2}-C\right) - i\left(\frac{1}{2}-S\right)\right\}}\right|^2$$

(The constants in A_1 and A_2 must be the same by symmetry)

In our case, $a = 0{\cdot}30\text{ mm}$, $\lambda = 0{\cdot}65\ \mu\text{m}$, $b = 1{\cdot}1\text{ m}$

$$\nu = 0{\cdot}30 \times \sqrt{\frac{2}{1{\cdot}1 \times 0{\cdot}65}} = 0{\cdot}50$$

$$C(0{\cdot}50) = 0{\cdot}48 \qquad S(0{\cdot}50) = 0{\cdot}06$$

$$\frac{I_2}{I_1} = \left| \frac{0{\cdot}02 - 0{\cdot}44\, i + e^{-i\delta}\dfrac{(1-i)}{2}}{\dfrac{1-i}{2} e^{i\delta} + 0{\cdot}02 - 0{\cdot}44\, i} \right|^2 = \left| \frac{1 + (0{\cdot}02 - 0{\cdot}44\, i)\sqrt{2}\, e^{i\delta + \frac{i\pi}{4}}}{1 + (0{\cdot}02 - 0{\cdot}44\, i)\sqrt{2}\, e^{-i\delta + \frac{i\pi}{4}}} \right|^2$$

But $0{\cdot}02 - 0{\cdot}44\, i = 0{\cdot}44\, e^{i\alpha}$, $\alpha = 1{\cdot}525$ rad ($\approx 87{\cdot}4°$)

$$\text{So } \frac{I_2}{I_1} = \left| \frac{1 + 0{\cdot}44 \times \sqrt{2} \times e^{i(\delta - 0{\cdot}740)}}{1 + 0{\cdot}44 \times \sqrt{2} \times e^{-i(\delta + 0{\cdot}740)}} \right|^2 = \frac{1 + 2(0{\cdot}44)^2 + 2\sqrt{2} \times 0{\cdot}44 \cos(\delta - 0{\cdot}740)}{1 + 2(0{\cdot}44)^2 + 2\sqrt{2} \times 0{\cdot}44 \cos(\delta + 0{\cdot}740)}$$

I_2 is maximum when $\quad \delta - 0{\cdot}740 = 0$ (modulo 2π)

$$\text{Thus in that case} \quad \frac{I_2}{I_1} = \frac{1{\cdot}387 + 1{\cdot}245}{1{\cdot}387 + 1{\cdot}245 \cos(1{\cdot}48)} = \frac{2{\cdot}632}{1{\cdot}5} \approx 1{\cdot}75$$

Q.116. A plane monochromatic light wave of intensity I_o falls normally on an opaque screen with a long slit having a semicircular

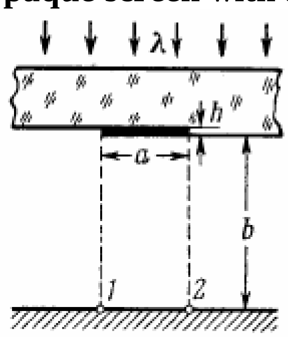


Fig. 5.23.

cut on one side (Fig. 5.24). The edge of the cut coincides with the boundary line of the first Fresnel zone for the observation point P. The width of the slit measures 0.90 of the radius of the cut. Using Fig. 5.19, find the intensity of light at the point P.

Ans. $\displaystyle \int_{aperture} \frac{a_0}{r} e^{-ikr}\, dS = \int_{Semicircle} + \int_{Slit}$

The contribution of the full 1st Fresnel zone has been evaluated in 5.103. The contribution of the semi-circle is one half of it and is

$$-\frac{2\pi}{k}i a_0 e^{-ikb} = -i a_0 \lambda e^{-ikb}$$

The contribution of the slit is

$$\frac{a_0}{b}\int_0^{0\cdot 90\sqrt{b\lambda}} e^{-ikb}\, e^{-ik\frac{x^2}{2b}}\, dx \int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy$$

Now
$$\int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy = \int_{-\infty}^{\infty} e^{-i\frac{\pi y^2}{b\lambda}}\, dy$$

$$\sqrt{\frac{b\lambda}{2}}\int_{-\infty}^{\infty} e^{-i\pi u^2/2}\, du = \sqrt{b\lambda}\; e^{-i\pi/4}$$

Thus the contribution of the slit is

$$\frac{a_0}{b}\sqrt{b\lambda}\; e^{-ikb-i\pi/4}\int_0^{0\cdot 9\times\sqrt{2}} e^{-i\pi u^2/2}\, du\sqrt{\frac{b\lambda}{2}}$$

$$= a_0\lambda e^{-ikb-i\pi/4}\frac{1}{\sqrt{2}}\int_0^{1\cdot 27} e^{-i\pi u^2/2}\, du$$

Thus the intensity at the observation point P on the screeii is

$$a_0^2\lambda^2\left|-i+\frac{1-i}{2}(C(1\cdot 27)-iS(1\cdot 27))\right|^2 = a_0^2\lambda^2\left|-i+\frac{(1-i)(0\cdot 67-0\cdot 65\, i)}{2}\right|^2$$

(on using $C(1\cdot 27) = 0\cdot 67$ and $S(1\cdot 27) = 0\cdot 65$)

$$= a_0^2\lambda^2\,|-i+0\cdot 01-0\cdot 66\, i|^2$$

$$= a_0^2\lambda^2\,|0\cdot 01-1\cdot 66\, i|^2$$

$$= 2\cdot 76\; a_0^2\lambda^2$$

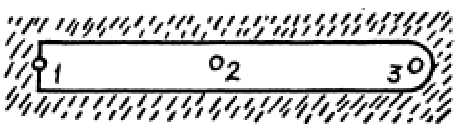
Now $a_0^2\lambda^2$ is the intensity due to half of 1st Fresnel zone and is therefore equal to I_0. (It can also be obtained by doing the x-integral over $-\infty$ to $+\infty$).

Thus $$I = 2\cdot 76\, I_0 .$$

Q.117. A plane monochromatic light wave falls normally on an opaque screen with a long slit whose shape is shown in Fig. 5.25. Making use of Fig. 5.19, find the ratio of intensities of light at points 1, 2, and 3 located behind the screen at equal distances from it. For point 3 the rounded-off edge of the slit coincides with the boundary line of the first Fresnel zone.

Ans. From the statement of the problem we know that the width of the slit = diameter of the first Fresnel zone $= 2\sqrt{b\lambda}$ where b is the distance of the observation point from the slit

We calculate the amplitudes by evaluating the integral of problem 5.103 we get



$$A_1 = \frac{a_0}{b} \int_{-\sqrt{b\lambda}}^{\sqrt{b\lambda}} e^{-ikb}\, e^{-ik\frac{x^2}{2b}}\, dx \int_0^{\infty} e^{-ik\frac{y^2}{2b}}\, dy$$

$$= \frac{a_0}{b} e^{-ikb} \frac{b\lambda}{2} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-i\pi u^2/2} du \times \int_0^{\infty} e^{-i\pi u^2/2} du$$

$$= \frac{a_0\lambda}{2}(1-i)e^{-ikb}\left(C\left(\sqrt{2}\right) - iS\left(\sqrt{2}\right)\right)$$

$$A_2 = \frac{a_0}{b} \int_{-\sqrt{b\lambda}}^{\sqrt{b\lambda}} e^{-ikb}\, e^{-ik\frac{x^2}{2b}} dx \int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy$$

$$= 2A_1$$

$$A_3 = -i a_0 \lambda e^{-ikb} + \frac{a_0\lambda(1-i)}{2}\left(C\left(\sqrt{2}\right) - iS\left(\sqrt{2}\right)\right)e^{-ikb}$$

Where the contribution of the 1st half Fresnel zone (in A3, first term) has been obtained from the last problem.

Thus

Thus $$I_1 = a_0^2\lambda^2 \left| \frac{(1-i)(0{\cdot}53 - 0{\cdot}72\, i)}{2} \right|^2$$

(on using $C\left(\sqrt{2}\right) = 0{\cdot}53,\ S\left(\sqrt{2}\right) = 0{\cdot}72$)

$$= a_0^2\lambda^2\, |-0{\cdot}095 - 0{\cdot}625\, i|^2 = 0{\cdot}3996\, a_0^2\lambda^2$$

$$I_2 = 4I_1$$

$$I_3 = a_0^2 \lambda^2 \mid -0{\cdot}095 - 0{\cdot}625\, i - i \mid^2$$
$$= a_0^2 \lambda^2 \mid -0{\cdot}095 - 1{\cdot}625\, i \mid^2$$
$$= 2{\cdot}6496\, a_0^2 \lambda^2$$

So $I_3 = 6{\cdot}6\, I_1$

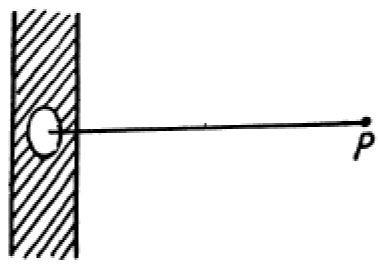
Thus $I_1 : I_2 : I_3 = 1 : 4 : 7$

Q.118. A plane monochromatic light wave falls normally on an opaque screen shaped as a long strip with a round hole in the middle. For the observation point P the hole corresponds to half the Fresnel zone, with the hole diameter being η = 1.07 times less than the width of the strip. Using Fig. 5.19, find the intensity of light at the point P provided that the intensity of the incident light is equal to I_0

Ans.

The radius of the first half Fresnel zone is $\sqrt{b\lambda/2}$ and the amplitude at P is obtained using problem 5.103.

$$A = \frac{a_0}{b}\left[\int_{-\infty}^{-\eta\sqrt{b\lambda/2}} + \int_{\eta\sqrt{b\lambda/2}}^{\infty}\right] e^{-ikb - \frac{kx^2}{2b}}\, dx$$



$$\int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy + \frac{a_0}{b} e^{-ikb} \int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\, 2\pi\rho\, d\rho .$$

We use

$$\int_{-\infty}^{-\eta\sqrt{b\lambda/2}} e^{-ikx^2/2b}\, dx$$
$$= \int_{\eta\sqrt{b\lambda/2}}^{\infty} e^{-ikx^2/2b}\, dx = \int_{\eta\sqrt{b\lambda/2}}^{\infty} e^{-i\frac{\pi x^2}{b\lambda}}\, dx$$

$$= \int_{\eta}^{\infty} e^{-i\pi u^2/2} \sqrt{\frac{b\lambda}{2}}\, du = \sqrt{\frac{b\lambda}{2}} \left(\int_0^{\infty} - \int_0^{\eta} \right) e^{-i\pi u^2/2}\, du.$$

$$= \sqrt{\frac{b\lambda}{2}} \left(\left(\frac{1}{2} - C(\eta) \right) - i \left(\frac{1}{2} - S(\eta) \right) \right)$$

Thus

$$A = a_0 \frac{\lambda}{\cancel{2}} \times \cancel{2} \times (1-i) e^{-ikb} \left[\left(\frac{1}{2} - C(\eta) \right) - i \left(\frac{1}{2} - S(\eta) \right) \right] + a_0 \lambda (1-i)\, e^{-ikb}$$

Where we have used

$$\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\, 2\pi\rho\, d\rho = \frac{2\pi i b}{k}(-1-i) = \frac{2\pi b}{k}(1-i) = \lambda b (1-i)$$

Thus the intensity is

$$I = |A|^2 = a_0^2 \lambda^2 \times 2 \left[(3/2 - C(\eta))^2 + \left(\frac{1}{2} - S(\eta) \right)^2 \right]$$

From Cornu’s Spiral,

$$C(\eta) = C(1{\cdot}07) = 0{\cdot}76$$
$$S(\eta) = S(1{\cdot}07) = 0{\cdot}50$$
$$I = a_0^2 \lambda^2 \times 2 \times (0{\cdot}74)^2 = 1{\cdot}09\, a_0^2 \lambda^2$$

As before

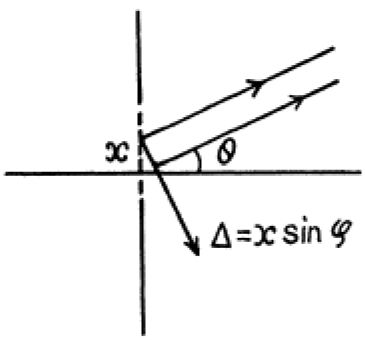
$I_0 = a_0^2 \lambda^2$ so $I = I_0$.

Diffraction of Light (Part - 2)

Q.119. Light with wavelength λ falls normally on a long rectangular slit of width b. Find the angular distribution of the intensity of light in the case of Fraunhofer diffraction, as well as the angular position of minima.

Ans.

If a plane wave is incident normally from the left on a slit of width b and the diffracted wave is observed at a large distance, the resulting pattern is called Fraunhofer diffraction. The condition for this is $b^2 \ll l\lambda$ where l is the distance between the slit and the screen. In practice light may be focussed on the screen with the help of a lens (or a telescope).



Consider an element of the slit which is an infinite strip of width dx. We use the formula of problem 5.103 with the following modifications.

The factor $\frac{1}{r}$ characteristic of spherical waves will be omitted. The factor $K(\varphi)$ will also be dropped if we confine overselves to not too large φ. In the direction defined by the angle φ the extra path difference of the wave emitted from the element at x relative to the wave emitted from the centre is

$$\Delta = -x \sin \varphi$$

Thus the amplitude of the wave is given by

$$\alpha \int_{-b/2}^{+b/2} e^{ik\sin\varphi}\,dx = \left(e^{i\frac{1}{2}kb\sin\varphi} - e^{-i\frac{1}{2}kb\sin\varphi}\right) \Big/ ik\sin\varphi$$

$$= \frac{\sin\left(\dfrac{\pi b}{\lambda}\sin\varphi\right)}{\dfrac{\pi b}{\lambda}\sin\varphi}$$

Thus $$I = I_0 \frac{\sin^2\alpha}{\alpha^2}$$

where $$\alpha = \frac{\pi b}{\lambda}\sin\varphi \text{ and}$$

I_0 is a constant

Minima are observed for $\sin\alpha = 0$ but $\alpha \neq 0$

Thus we find minima at angles given by

$$b\sin\varphi = k\lambda, \; k = \pm 1, \pm 2, \pm 3, \ldots$$

Q.120. Making use of the result obtained in the foregoing problem, find the conditions defining the angular position of maxima of the first, the second, and the third order.

Ans.

Since $I(\alpha)$ is *+ve* and vanishes for $b\sin\varphi = k\lambda$ i.e for $\alpha = k\pi$, we expect maxima of $I(\alpha)$ between $\alpha = +\pi$ & $\alpha = +2\pi$, etc. We can get these values by.

$$\frac{d}{d\alpha}(I(\alpha)) = I_0\, 2\,\frac{\sin\alpha}{\alpha}\,\frac{d}{d\alpha}\,\frac{\sin\alpha}{\alpha} = 0$$

$$\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = 0 \quad \text{or} \quad \tan\alpha = \alpha$$

Solutions of this transcendental equation can be obtained graphically.
The first three solutions are

$$\alpha_1 = 1{\cdot}43\,\pi, \; \alpha_2 = 2{\cdot}46\,\pi, \; \alpha_3 = 3{\cdot}47\,\pi$$

on the +ve side. (On the negative side the solution are $-\alpha_1, -\alpha_2, -\alpha_3, \ldots$)

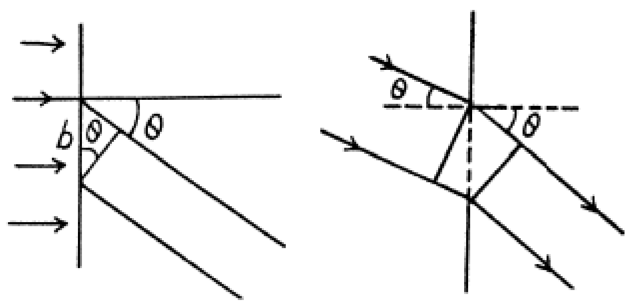
Thus
$$b\sin\varphi_1 = 1{\cdot}43\,\lambda$$
$$b\sin\varphi_2 = 2{\cdot}46\,\lambda$$
$$b\sin\varphi_3 = 3{\cdot}47\,\lambda$$

Asymptotically the solutions are

$$b\sin\varphi_m = \left(M + \frac{1}{2}\right)\lambda$$

Q.121. Light with wavelength λ = 0.50 [tm falls on a slit of width b = 10 um at an angle θ_0 = 30° to its normal. Find the angular position of the first minima located on both sides of the central Fraunhofer maximum.

Ans. The relation b sinθ = kλ for minima (when light is incident normally on the slit) has a simple interpretation $: b \sin\theta$ is the path difference between extreme wave normals emitted at angle θ



When light is incident at an angle θ_0 the path difference is

$$b(\sin\theta - \sin\theta_0)$$

and the condition of minima is

$$b(\sin\theta - \sin\theta_0) = k\lambda$$

For the first minima

$$b(\sin\theta - \sin\theta_0) = \pm\lambda \quad \text{or} \quad \sin\theta = \sin\theta_0 \pm \frac{\lambda}{b}$$

Putting in numbers $\theta_0 = 30°$, $\lambda = 0{\cdot}50\ \mu\text{m}$ $b = 10\ \mu\text{m}$

$$\sin\theta = \frac{1}{2} \pm \frac{1}{20} = 0{\cdot}55 \quad \text{or} \quad 0{\cdot}45$$

$$\theta_{+1} = 33° - 20' \quad \text{and} \quad \theta_{-1} = 26°\,44'$$

Q.122. A plane light wave with wavelength λ = 0.60 ti,m falls normally on the face of a glass wedge with refracting angle $\Theta = 15°$. The opposite face of the wedge is opaque and has a slit of width b = 10 µm parallel to the edge. Find:
(a) the angle $\Delta\theta$ between the direction to the Fraunhofer maximum of zeroth order and that of incident light;
(b) the angular width of the Fraunhofer maximum of the zeroth order.

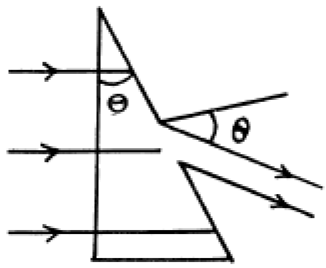
Ans.

(a) This case is analogous to the previous one except that the incident wave moves in glass of RI n. Thus the expression for the path difference for light diffracted at angle θ from the normal to the hypotenuse of the wedge is

$$b(\sin\theta - n\sin\Theta)$$

we write $\theta = \Theta + \Delta\theta$

Then for the direction of principal Fraunhofer maximum



$$b(\sin(\Theta + \Delta\theta) - n\sin\Theta) = 0$$

or $\Delta\theta = \sin^{-1}(n\sin\Theta) - \Theta$

Using $\Theta = 15°$, $n = 1{\cdot}5$ we get

$$\Delta\theta = 7{\cdot}84°$$

(b) The width of the central maximum is obtained from $(\lambda = 0{\cdot}60\ \mu m,\ b = 10\ \mu m)$

$$b(\sin\theta_1 - n\sin\Theta) = \pm\lambda$$

Thus

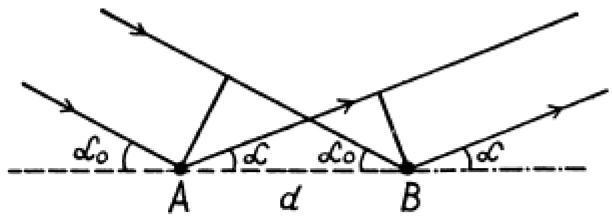
$$\theta_{+1} = \sin^{-1}\left(n\sin\Theta + \frac{\lambda}{b}\right) = 26{\cdot}63°$$

$$\theta_{-1} = \sin^{-1}\left(n\sin\Theta \frac{\lambda}{b}\right) = 19{\cdot}16°$$

$$\therefore\quad \delta\theta = \theta_{+1} - \theta_{-1} = 7{\cdot}47°$$

Q.123. A monochromatic beam falls on a reflection grating with period d = 1.0 mm at a glancing angle α_0 = 1.0°. When it is diffracted at a glancing angle α = 3.0° a Fraunhofer maximum of second order occurs. Find the wavelength of light.

Ans.



The path difference between waves reflected at A and B is

$$d(\cos\alpha_0 - \cos\alpha)$$

and for maxima

$$d(\cos\alpha_0 - \cos\alpha) = k\lambda, \quad k = 0, \pm 1, \pm 2, \ldots$$

In our case, $k = 2$ and α_0, α are small in radiaus. Then

$$2\lambda = d\left(\frac{\alpha^2 - \alpha_0^2}{2}\right)$$

Thus $$\lambda = \frac{(\alpha^2 - \alpha_0^2)\,d}{4} = 0{\cdot}61\ \mu\text{m}$$

for $$\alpha = \frac{3\pi}{180},\ \alpha_0 = \frac{\pi}{180},\ d = 10^{-3}\,\text{m}$$

Q.124. Draw the approximate diffraction pattern originating in the case of the Fraunhofer diffraction from a grating consisting of three identical slits if the ratio of the grating period to the slit width is equal to (a) two; (b) three.

Ans. The general formula for diffraction from N slits is

$$I = I_0\,\frac{\sin^2\alpha}{\alpha^2}\,\frac{\sin^2 N\beta}{\sin^2\beta}$$

$$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

Where

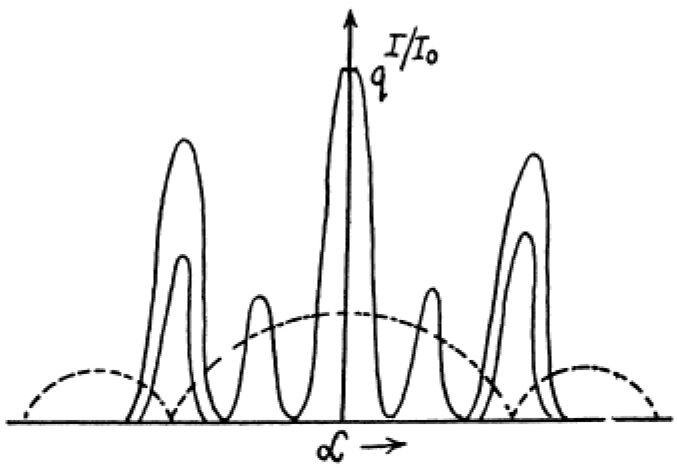
$$\beta = \frac{\pi(a+b)\sin\theta}{\lambda}$$

and $N = 3$ in the cases here.

(a) In this case $a + b = 2a$

so $$\beta = 2\alpha \quad\text{and}\quad I = I_0\,\frac{\sin^2\alpha}{\alpha^2}\,(3 - 4\sin^2 2\alpha)^2$$

On plotting we get a curve that qualitatively looks like the one below



(b) In this case $a + b = 3a$

so $$\beta = 3\alpha$$

and $$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}(2 - 4\sin^2 3\alpha)^2$$

This has 3 minima between the principal maxima

Q.125. With light falling normally on a diffraction grating, the angle of diffraction of second order is equal to 45° for a wavelength λ_1 = 0.65 µm. Find the angle of diffraction of third order for a wave length λ_2 = 0 .50 1.1.m.

Ans.

From the formula $d \sin \theta = m\lambda$

we have $d \sin 45° = 2\lambda_1 = 2 \times 0{\cdot}65\ \mu m$

or $$d = 2\sqrt{2} \times 0{\cdot}65\ \mu m$$

Then for $\lambda_2 = 0{\cdot}50$ in the third order

$$2\sqrt{2} \times 0{\cdot}65 \sin \theta = 3 \times 0{\cdot}50$$

$$\sin \theta = \frac{1{\cdot}5}{1{\cdot}3 \times \sqrt{2}} = 0{\cdot}81602$$

This gives $\theta = 54{\cdot}68° \approx 55°$

Q.126. Light with wavelength 535 nm falls normally on a diffraction grating. Find its period if the diffraction angle 35°corresponds to one of the Fraunhofer maxima and the highest order of spectrum is equal to five.

Ans. The diffraction formula is

$$d \sin \theta_0 = n_0 \lambda$$

where $\theta_0 = 35°$ is the angle of diffraction corresponding to order n_0 (which is not yet known).

Thus $$d = \frac{n_0 \lambda}{\sin \theta_0} = n_0 \times 0{\cdot}9327 \ \mu\text{m}$$

on using $\lambda = 0{\cdot}535 \ \mu\text{m}$

For the n^{th} order we get

$$\sin \theta = \frac{n}{n_0} \sin \theta_0 = \frac{n}{n_0}(0{\cdot}573576)$$

If $n_0 = 1$, then $n > n_0$ is at least 2 and $\sin \theta > 1$ so $n = 1$ is the highest order of diffraction.

If $n_0 = 2$ then $n = 3, 4$, but $\sin \theta > 1$ for $n = 4$ thus the highest order of diffraction is 3.

If $n_0 = 3$,

then $$n = 4, 5, 6.$$

For $n = 6$, $\sin \theta = 2 \times 0{\cdot}57 > 1$, so not allowed while for

$$n = 5, \ \sin \theta = \frac{5}{3} \times 0{\cdot}573576 < 1$$

is allowed. Thus in this case the highest order of diffraction is five as given. Hence

$$n_0 = 3$$

and $$d = 3 \times 0{\cdot}9327 = 2{\cdot}7981 \approx 2{\cdot}8 \ \mu\text{m}.$$

Q.127. Find the wavelength of monochromatic light falling normally on a diffraction grating with period d = 2.2 µm if the angle between the directions to the Fraunhofer maxima of the first and the second order is equal to Δθ = 15°.

Ans.

Given that

$$d \sin \theta_1 = \lambda$$

$$d \sin \theta_2 = d \sin(\theta_1 + \Delta\theta) = 2\lambda$$

Thus $$\sin \theta_1 \cos \Delta\theta + \cos \theta_1 \sin \Delta\theta = 2 \sin \theta_1$$

or $$\sin \theta_1 (2 - \cos \Delta\theta) = \cos \theta_1 \sin \Delta\theta$$

or $$\tan \theta_1 = \frac{\sin \Delta\theta}{2 - \cos \Delta\theta}$$

or $$\sin \theta_1 = \frac{\sin \Delta\theta}{\sqrt{\sin^2 \Delta\theta + (2 - \cos \Delta\theta)^2}} = \frac{\sin \Delta\theta}{\sqrt{5 - 4 \cos \Delta\theta}}$$

Finally $$\lambda = \frac{d \sin \Delta\theta}{\sqrt{5 - 4 \cos \Delta\theta}}.$$

Substitution gives $\lambda \approx 0{\cdot}534 \ \mu\text{m}$

Q.128. Light with wavelength 530 nm falls on a transparent diffraction grating with period 1.50 lila. Find the angle, relative to the grating normal, at which the Fraunhofer maximum of highest order is observed provided the light falls on the grating (a) at right angles; (b) at the angle 60° to the normal.

Ans. (a) Here the simple formula

$d \sin \theta = m_1 \lambda$ holds.

$1{\cdot}5 \sin \theta = m \times 0{\cdot}530 \quad \sin \theta = \frac{m \times 0{\cdot}530}{1{\cdot}5}$

Highest permissible m is $m = 2$ because $\sin \theta > 1$ if $m = 3$. Thus

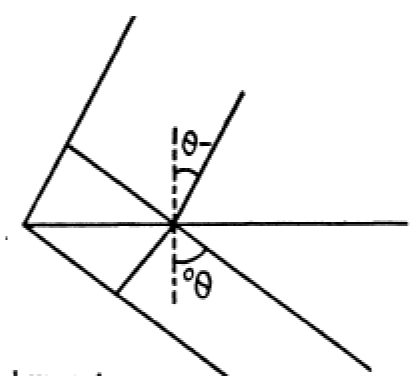
$$\sin \theta = \frac{1{\cdot}06}{1{\cdot}50} \text{ for } m = 2, \text{ This gives } \theta = 45^{\circ} \text{ nearby.}$$

(b) Here $d(\sin \theta_0 - \sin \theta) = n\lambda$

Thus $\sin \theta = \sin \theta_0 - \frac{n\lambda}{d}$

$$= \sin 60^{\circ} - n \times \frac{0{\cdot}53}{1{\cdot}5}$$

$$= 0{\cdot}86602 - n \times 0{\cdot}353333.$$



For $n = 5$, $\sin \theta = -0{\cdot}900645$

for $n = 6$, $\sin \theta < -1$.

Thus the highest order is $n = 5$ and we get

$\theta = \sin^{-1}(-0{\cdot}900645) \cong -64^{\circ}$

Q.129. Light with wavelength λ = 0.60 tim falls normally on a diffraction grating inscribed on a plane surface of a plano-convex cylindrical glass lens with curvature radius R = 20 cm. The period of the grating is equal to d = 6.0 µm. Find the distance between the principal maxima of first order located symmetrically in the focal plane of that lens.

Ans. For the Lens

$$\frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{\infty}\right) \quad \text{or} \quad f = \frac{R}{n-1}$$

For the grating

$$d \sin\theta_1 = \lambda \quad \text{or} \quad \sin\theta_1 = \frac{\lambda}{d}$$

$$\operatorname{cosec}\theta_1 = \frac{d}{\lambda}, \ \cot\theta_1 = \sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}$$

$$\tan\theta_1 = \frac{1}{\sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}}$$

Hence the distance between the two symmetrically placed first order maxima

$$= 2 f \tan\theta_1 = \frac{2R}{(n-1)\sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}}$$

On putting $R = 20,\ n = 1{\cdot}5,\ d = 6{\cdot}0\ \mu\text{m}$

$\lambda = 0{\cdot}60\ \mu\text{m}$ we get 8·04 cm.

Q.130. A plane light wave with wavelength λ = 0.50 µm falls normally on the face of a glass wedge with an angle $\Theta = 30^\circ$. On the opposite face of the wedge a transparent diffraction grating with period d = 2.00 µm is inscribed, whose lines are parallel to the wedge's edge. Find the angles that the direction of incident light forms with the directions to the principal Fraunhofer maxima of the zero and the first order. What is the highest order of the spect rum? At what angle to the direction of incident light is it observed?

Ans. The diffraction formula is easily obtained on taking account of the fact that the optical path in the glass wedge acquires a factor n (refractive index). We get

$$d\,(n \sin\Theta - \sin(\Theta\,\theta_k)) = k\lambda$$

Since $n > 0$, $\Theta - \theta_0 > \Theta$ and so θ_0 must be negative. We get, using $\Theta = 30^\circ$

$$\frac{3}{2} \times \frac{1}{2} = \sin(30^\circ - \theta_0) = \sin 48{\cdot}6^\circ$$

Thus $$\theta_0 = -18.6°$$

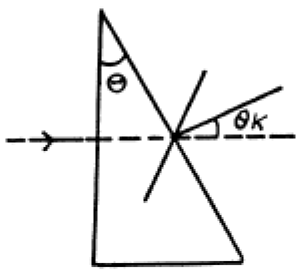
Also for $k = 1$

$$\frac{3}{4} - \sin(30° - \theta_{+1}) = \frac{\lambda}{d} = \frac{0.5}{2.0} = \frac{1}{4}$$

Thus $$\theta_{+1} = 0°$$

We calculate θ_k for various k by the above formula. For $k = 6$.

$$\sin(\theta_k - 30°) = \frac{3}{4} \Rightarrow \theta_k = 78.6°$$



For $k = 7$

$$\sin(\theta_k - 30°) = +1 \Rightarrow \theta_k = 120°$$

This is in admissible. Thus the highest order that can be observed is

$$k = 6$$

corresponding to $$\theta_k = 78.6°$$

(for $k = 7$ the diffracted ray will be grazing the wedge).

Q.131. A plane light wave with wavelength λ falls normally on a phase diffraction grating whose side view is shown in Fig. 5.26. The grating is cut on a glass plate with refractive index n. Find the depth h of the lines at which the intensity of the central Fraunhofer maximum is equal to zero. What is in this case the diffraction angle corresponding to the first maximum?

Ans. The intensity of the central Fraunhofer maximum will be zero if the waves from successive grooves (not in the same plane) differ in phase by an odd multiple of n. Then since the phase difference is

$$\delta = \frac{2\pi}{\lambda}(n-1)h$$

for the central ray we have

$$\frac{2\pi}{\lambda}(n-1)h = \left(k - \frac{1}{2}\right)2\pi, \ k = 1, 2, 3, \ldots$$

or $$h = \frac{\lambda}{n-1}\left(k - \frac{1}{2}\right).$$

The path difference between the rays 1 & 2 is approximately (neglecting terms of order θ^2)

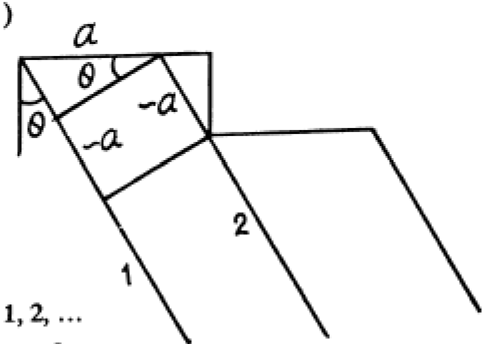
$a \sin\theta + a - n a$

$= a \sin\theta - (n-1) a$

Thus for a maximum

$$a \sin\theta - \left(k' + \frac{1}{2}\right)\lambda = m\lambda$$

or $a \sin\theta = \left(m + k' + \frac{1}{2}\right)\lambda,$



$$k' = 0, 1, 2, \dots$$
$$m = 0, \pm 1, \pm 2, \dots$$

The first maximum after the central minimum is obtained when $m + k' = 0$

We get $$a \sin\theta_1 = \frac{1}{2}\lambda$$

Q.132. Figure 5.27 illustrates an arrangement employed in observations of diffraction of light by ultrasound. A plane light wave with wavelength λ = 0.55 μm passes through the water-filled tank T in which a standing ultrasonic wave is sustained at a frequency ν = 4.7 MHz As a result of diffraction of light by the optically inhomogeneous periodic structure a diffraction spectrum can be observed in the focal plane of the objective O with focal length f = 35 cm. The separation between neighbouring maxima is Δx = 0.60 mm. Find the propagation velocity of ultrasonic oscillations in water.

Ans. When standing ultra-sonic waves are sustained in the tank it behaves like a grating whose grating element is

$$d = \frac{v}{\nu} = \text{wavelength of the ultrasonic}$$

v = velocity of ultrasonic. Thus for maxima

$$\frac{v}{\nu} \sin\theta_m = m\lambda$$

On the other hand

$$f \tan\theta_m = m\,\Delta x$$

Assuming θ_m to be small $\left(\text{because } \lambda << \frac{v}{\nu}\right)$

we get $$\Delta x = \frac{f \tan\theta_m}{m} = \frac{f \tan\theta_m}{\frac{v}{\nu\lambda}\sin\theta_m} = \frac{\lambda \nu f}{v}$$

or $$v = \frac{\lambda v f}{\Delta x}$$

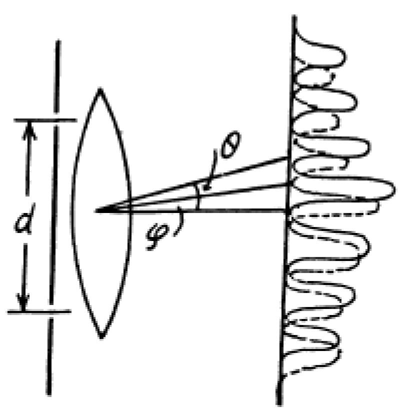
Putting the values $\lambda = 0{\cdot}55\ \mu m$, $v = 4{\cdot}7$ MHz

$f = 0{\cdot}35$ m and $\Delta x = 0{\cdot}60 \times 10^{-3}$ m we easily get

$$v = 1{\cdot}51 \text{ km/sec}.$$

Q.133. To measure the angular distance ψ between the components of a double star by Michelson's method, in front of a telescope's lens a diaphragm was placed, which had two narrow parallel slits separated by an adjustable distance d. While diminishing d, the first smearing of the pattern was observed in the focal plane of the objective at d = 95 cm. Find ψ assuming the wavelength of light to be equal to λ = 0.55 μm.

Ans. Each star produces its own diffraction pattern in the focal plane o f the objective and these patterns are separated by angle As the distance d decreases the angle 0 between the neighbouring maxima in either diffraction pattern increases (sinθ - λ/d). When θ becomes equal to 2ψ the first deterioration of visibility occurs because the maxima of one system of fringes coincide with the minima of the other system. Thus from the condition



$\theta = 2\psi$ and $\sin\theta = \frac{\lambda}{d}$ we get

$\psi = \frac{1}{2}\theta = \frac{\lambda}{2d}$ (radians)

Putting the values we get $\psi = 0{\cdot}06''$

Q.134. A transparent diffraction grating has a period d = 1.50 μm. Find the angular dispersion D (in angular minutes per nanometers) corresponding to the maximum of highest order for a spectral line of wavelength λ = 530 nm of light falling on the grating (a) at right angles; (b) at the angle θ_0 = 45° to the normal.

Ans.

(a) For normal incidence, the maxima are given by

$$d \sin\theta = n\lambda$$

so $$\sin\theta = n\frac{\lambda}{d} = n \times \frac{0{\cdot}530}{1{\cdot}500}$$

Clearly $n \le 2$ as $\sin\theta > 1$ for $n = 3$.

Thus the highest order is $n = 2$. Then

$$D = \frac{d\theta}{d\lambda} = \frac{k}{d\cos\theta} = \frac{k}{d}\frac{1}{\sqrt{1-\left(\frac{k\lambda}{d}\right)^2}}$$

Putting $k = 2$, $\lambda = 0{\cdot}53\,\mu\text{m}$, $d = 1{\cdot}5\,\mu\text{m} = 1500\,\text{nm}$

we get $D = \dfrac{2}{1500}\dfrac{1}{\sqrt{1-\left(\frac{1{\cdot}06}{1{\cdot}5}\right)^2}} \times \dfrac{180}{\pi} \times 60 = 6{\cdot}47$ ang. min/nm.

(b) We write the diffraction formula as

$$d(\sin\theta_0 + \sin\theta) = k\lambda$$

so $$\sin\theta_0 + \sin\theta = k\frac{\lambda}{d}$$

Here $$\theta_0 = 45° \text{ and } \sin\theta_0 = 0{\cdot}707$$

so $\sin\theta_0 + \sin\theta \le 1{\cdot}707$. Since

$\dfrac{\lambda}{d} = \dfrac{0{\cdot}53}{1{\cdot}5} = 0{\cdot}353333$, we see that

$$k \le 4$$

Thus highest order corresponds to $k = 4$.

Now as before $D = \dfrac{d\theta}{d\lambda}$ so

$$D = \frac{k}{d\cos\theta} = \frac{k/d}{\sqrt{1-\left(\frac{k\lambda}{d} - \sin\theta_0\right)^2}}$$

$$= 12{\cdot}948 \text{ ang. min/nm},$$

Q.135. Light with wavelength λ falls on a diffraction grating at right angles. Find the angular dispersion of the grating as a function of diffraction angle θ.

Ans.

We have $$d \sin \theta = k \lambda$$

so $$\frac{d\theta}{d\lambda} = D = \frac{k}{d \cos \theta} = \frac{\tan \theta}{\lambda}$$

Q.136. Light with wavelength λ = 589.0 nm falls normally on a diffraction grating with period d = 2.5 µm, comprising N = = 10 000 lines. Find the angular width of the diffraction maximum of second order.

Ans.

For the second order principal maximum

$$d \sin \theta_2 = 2\lambda = k\lambda$$

or $$\frac{N\pi}{\lambda} d \sin \theta_2 = 2N\pi$$

minima adjacent to this maximum occur at

$$\frac{N\pi}{\lambda} d \sin(\theta_2 \pm \Delta\theta) = (2N \pm 1)\pi$$

Or $$d \cos \theta_2 \Delta\theta = \frac{\lambda}{N}$$

Finally angular width of the 2nd principal maximum is

$$2\Delta\theta = \frac{2\lambda}{N d \cos \theta_2} = \frac{2\lambda}{N d \sqrt{1 - (k\lambda/d)^2}} = \frac{\tan \theta_2}{N}$$

On putting the values we get 11·019″ of arc

Q.137. Demonstrate that when light falls on a diffraction grating at right angles, the maximum resolving power of the grating cannot exceed the value l/λ where l is the width of the grating and λ is the wavelength of light.

Ans. Using

$$R = \frac{\lambda}{\delta\lambda} = kN = \frac{N d \sin \theta}{\lambda}$$

$$= \frac{l \sin \theta}{\lambda} \leq \frac{l}{\lambda}.$$

Diffraction of Light (Part - 3)

Q.138. Using a diffraction grating as an example, demonstrate that the frequency difference of two maxima resolved according to Rayleigh's criterion is equal to the reciprocal of the difference of propagation times of the extreme interfering oscillations, i.e. δv = = 1/δt.

Ans. For the just resolved waves the frequency difference

$$\delta \nu = \frac{c\,\delta\lambda}{\lambda} = \frac{c}{\lambda R} = \frac{c}{\lambda k N}$$

$$= \frac{c}{N d \sin\theta} = \frac{1}{\delta t}$$

since N d sinθ is the path difference between waves emitted by the extremities o f the grating.

Q.139. Light composed of two spectral lines with wavelengths 600.000 and 600.050 nm falls normally on a diffraction grating 10.0 mm wide. At a certain diffraction angle θ these lines are close to being resolved (according to Rayleigh's criterion). Find θ.

Ans.

$\delta\lambda = \cdot 050$ nm

$$R = \frac{\lambda}{\delta\lambda} = \frac{600}{\cdot 05} = 12000 \text{ (nearly)}$$

$$= kN$$

On the other hand

$$d \sin\theta = k\lambda$$

Thus

$$\frac{l}{kN}\sin\theta = \lambda$$

where $l = 10^{-2}$ metre is the width of the grating

Hence

$$\sin\theta = 12000 \times \frac{\lambda}{l}$$

$$= 12000 \times 600 \times 10^{-7} = 0{\cdot}72$$

or

$$\theta = 46^\circ .$$

Q.140. Light falls normally on a transparent diffraction grating of width l = 6.5 cm with 200 lines per millimeter. The spectrum under investigation includes a spectral

line with λ = 670.8 nm consisting of two components differing by δλ = 0.015 nm. Find: (a) in what order of the spectrum these components will be resolved; (b) the least difference of wavelengths that can be resolved by this grating in a wavelength region λ ≈ 670 nm.

Ans. We see that

$$N = 6{\cdot}5 \times 10 \times 200 = 13000$$

Now to resolve lines with $\delta\lambda = 0{\cdot}015$ nm and $\lambda \approx 670{\cdot}8$ nm we must have

$$R = \frac{670{\cdot}8}{0{\cdot}015} = 44720$$

Since $3N < R < 4N$ one must go to the fourth order to resolve the said components.

(b) we have $d = \frac{1}{200}$ mm $= 5\,\mu$m

so $$\sin\theta = \frac{k\lambda}{d} = \frac{k \times 0{\cdot}670}{5}$$

since $|\sin\theta| \le 1$ we must have $k \le 7{\cdot}46$

so $$k_{max} = 7 \approx \frac{d}{\lambda}$$

Thus $$R_{max} = k_{max}\,N = 91000 \approx \frac{Nd}{\lambda} = \frac{l}{\lambda}$$

where $l = 6{\cdot}5$ cm is the grating width.

Finally $$\delta\lambda_{min} = \frac{\lambda}{R_{max}} = \frac{670}{91000} = {\cdot}007\text{ nm} = 7\text{ pm} \approx \frac{\lambda^2}{l}.$$

Q.141. With light falling normally on a transparent diffraction grating 10 mm wide, it was found that the components of the yellow line of sodium (589.0 and 589.6 nm) are resolved beginning with the fifth order of the spectrum. Evaluate:
(a) the period of this grating;
(b) what must be the width of the grating with the same period for a doublet λ = 460.0 nm whose components differ by 0.13 nm to be resolved in the third order of the spectrum.

Ans. Here

$$R = \frac{\lambda}{\delta\lambda} = \frac{589{\cdot}3}{0{\cdot}6} = \text{kN} = 5\text{N}$$

So

$$N = \frac{589 \cdot 3}{3} = \frac{10^{-2}}{d}$$

$$d = \frac{3 \times 10^{-2}}{589 \cdot 3} \text{m} = \cdot 0509 \text{ mm}$$

(b) To resolve a doublet with $\lambda = 460 \cdot 0 \text{ nm and } \delta\lambda = 0 \cdot 13 \text{ nm}$ in the third order we must have

$$N = \frac{R}{3} = \frac{460}{3 \times 0 \cdot 13} = 1179$$

This means that the grating is

$$N d = 1179 \times 0 \cdot 0509 = 60 \cdot 03 \text{ mm}$$

wide = 6cm wide.

Q.142. A transparent diffraction grating of a quartz spectrograph is 25 mm wide and has 250 lines per millimeter. The focal length of an objective in whose focal plane a photographic plate is located is equal to 80 cm. Light falls on the grating at right angles. The spectrum under investigation includes a doublet with components of wavelengths 310.154 and 310.184 nm. Determine:
(a) the distances on the photographic plate between the components of this doublet in the spectra of the first and the second order;
(b) whether these components will be resolved in these orders of the spectrum.

Ans.

(a) From $d \sin\theta = k\lambda$

we get $$\delta\theta = \frac{k\,\delta\lambda}{d\cos\theta}$$

On the other hand $$x = f\sin\theta$$

so $$\delta x = f\cos\theta\,\delta\theta = \frac{kf}{d}\delta\lambda$$

For $$f = 0 \cdot 80 \text{ m},\ \delta\lambda = 0 \cdot 03 \text{ nm, and}$$

$$d = \frac{1}{250} \text{mm}$$

we get $$\delta x = \begin{cases} 6\,\mu\text{m} & \text{if } k = 1 \\ 12\,\mu\text{m} & \text{if } k = 2 \end{cases}$$

(b) Here $N = 25 \times 250 = 6250$

and $$\frac{\lambda}{\delta \lambda} = \frac{310 \cdot 169}{0 \cdot 03} = 10339 \cdot\cdot > N$$

and so to resolve we need $k = 2$ For $k = 1$ gives an R.P. of only 6250.

Q.143. The ultimate resolving power $\lambda\delta\lambda$ of the spectrograph's trihedral prism is determined by diffraction of light at the prism edges (as in the case of a slit). When the prism is oriented to the least deviation angle in accordance with Rayleigh's criterion,

$$\lambda/\delta\lambda = b \mid dn/d\lambda \mid,$$

where b is the width of the prism's base (Fig. 5.28), and dnIdk is the dispersion of its material. Derive this formula.

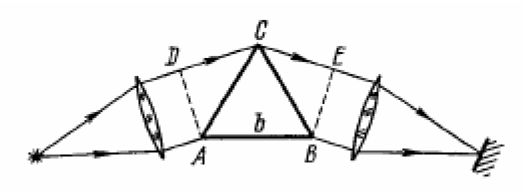
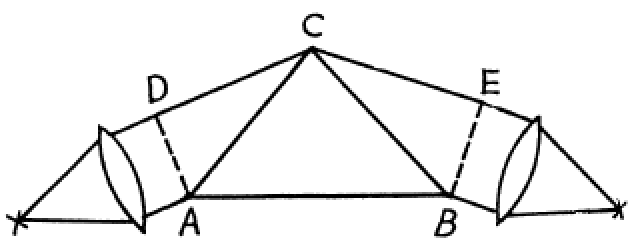


Fig. 5.28.

Ans. Suppose the incident light consists of two wavelengths λ and $\lambda + \delta\lambda$ which are just resolved by the prism. Then by Rayleigh's criterion, the maximum of the line of wavelength λ must coincide with the first minimum of the line of wavelength $\lambda + \delta\lambda$, Let us write both conditions in terms of the optical path differences for the extreme rays : For the light of wavelength λ



$$bn - (DC + CE) = 0$$

For the light of wavelength $\lambda + \delta \lambda$

$$b(n + \delta n) - (DC + CE) = \lambda + \delta \lambda$$

because the path difference between extreme rays equals λ for the first minimum in a single slit diffraction (from the formula $a \sin \theta = \lambda$).

Hence $$b \, \delta n = \lambda$$

and $$R = \frac{\lambda}{\delta \lambda} = b \left| \frac{\delta n}{\delta \lambda} \right| = b \left| \frac{dn}{d\lambda} \right|$$

Q.144. A spectrograph's trihedral prism is manufactured from glass whose refractive index varies with wavelength as $n = A + B/\lambda^2$ where A and B are constants, with B being equal to 0.010 1μm². Making use of the formula from the foregoing problem, find:
(a) how the resolving power of the prism depends on k; calculate the value of λδλ in the vicinity of λ_1 = 434 nm and λ_2 = 656 nm if the width of the prism's base is b = 5.0 cm;
(b) the width of the prism's base capable of resolving the yellow doublet of sodium (589.0 and 589.6 nm).

Ans.

(a) $\frac{\lambda}{\delta\lambda} = R = b\left|\frac{dn}{d\lambda}\right| = 2Bb/\lambda^3$

For $b = 5\text{ cm},\ B = 0{\cdot}01\ \mu\text{m}^2 \qquad \lambda_1 = 0{\cdot}434\ \mu\text{m} = 5\times10^4\ \mu\text{m}$

$$R_1 = 1{\cdot}223\times10^4$$

for

$$\lambda_2 = 0{\cdot}656\ \mu\text{m}$$

$$R_2 = 0{\cdot}3542\times10^4$$

(b) To resolve the D-lines we require

$$R = \frac{5893}{6} = 982$$

Thus

$$982 = \frac{0{\cdot}02\times b}{(0{\cdot}5893)^3}$$

$$b = \frac{982\times(0{\cdot}5893)^3}{0{\cdot}02}\ \mu\text{m} = 1{\cdot}005\times10^4\ \mu\text{m} = 1{\cdot}005\text{ cm}$$

Q.145. How wide is the base of a trihedral prism which has the same resolving power as a diffraction grating with 10 000 lines in the second order of the spectrum if $|dn/d\lambda| = 0.10\ \mu\text{m}^{-1}$?

Ans.

$b\left|\frac{dn}{d\lambda}\right| = kN = 2\times10{,}000$

$$b\times0{\cdot}10\ \mu\text{m}^{-1} = 2\times10^4$$

$$b = 2\times10^5\ \mu\text{m} = 0{\cdot}2\ m = 20\text{ cm}.$$

Q.146. There is a telescope whose objective has a diameter D = 5.0 cm. Find the resolving power of the objective and the minimum separation between two points at a distance l = 3.0 km from the telescope, which it can resolve (assume λ = 0.55 μm).

Ans. Resolving power of the objective

$$= \frac{D}{1 \cdot 22\,\lambda} = \frac{5 \times 10^{-2}}{1 \cdot 22 \times 0 \cdot 55 \times 10^{-6}} = 7 \cdot 45 \times 10^{4}$$

Let $(\Delta y)_{min}$ be the minimum distance between two points at a distance of 3.0 km which the telescope can resolve. Then

$$\frac{(\Delta y)_{min}}{3 \times 10^{3}} = \frac{1 \cdot 22\,\lambda}{D} = \frac{1}{7 \cdot 45 \times 10^{4}}$$

$$(\Delta y)_{min} = \frac{3 \times 10^{3}}{7 \cdot 45 \times 10^{4}} = 0 \cdot 04026 \text{ m} = 4 \cdot 03 \text{ cm}.$$

Q.147. Calculate the minimum separation between two points on the Moon which can be resolved by a reflecting telescope with mirror diameter 5 m. The wavelength of light is assumed to be equal to λ = 0.55 µm.

Ans. The limit of resolution of a reflecting telescope is determined by diffraction from the mirror and obeys a formula similar to that from a refracting telescope. The limit of resolution is

$$\frac{1}{R} = \frac{1 \cdot 22\,\lambda}{D} = \frac{(\Delta y)_{min}}{L}$$

where L = distance between the earth and the moon = 384000 km

Then putting the values $\lambda = 0 \cdot 55\ \mu\text{m}$, $D = 5$ m

we get $(\Delta y)_{min} = 51 \cdot 6$ metre

Q.148. Determine the minimum multiplication of a telescope with diameter of objective D = 5.0 cm with which the resolving power of the objective is totally employed if the diameter of the eye's pupil is d_o = 4.0 mm.

Ans. By definition, the magnification

$$\Gamma = \frac{\text{angle subtended by the image at the eye}}{\text{angle subtended by the object at the eye}} = \frac{\psi'}{\psi}$$

At the limit of resolution $\psi = \frac{1 \cdot 22\,\lambda}{D}$

where D = diameter of the objective

On the other hand to be visible to the eye $\psi' \geq \frac{1 \cdot 22\,\lambda}{d_0}$

where d_0 = diameter of the pupil

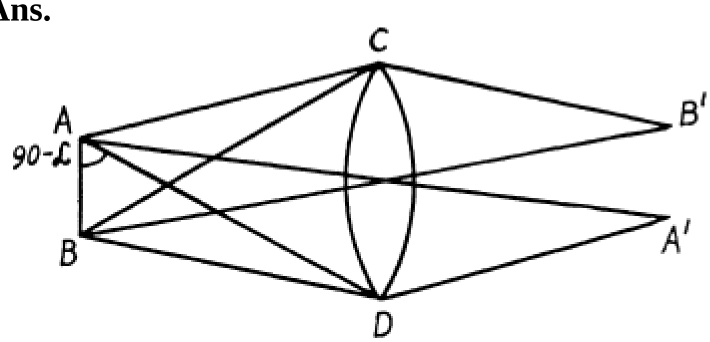
Thus to avail of the resolution offered by the telescope we must have

$$\Gamma \geq \frac{1{\cdot}22\,\lambda}{d_0} \Big/ \frac{1{\cdot}22\,\lambda}{D} = \frac{D}{d_0}$$

$$\Gamma_{min} = \frac{D}{d_0} = \frac{50\text{ mm}}{4\text{ mm}} = 12{\cdot}5$$

Q.149. There is a microscope whose objective's numerical aperture is sinα = 0.24, where α is the half-angle subtended by the objective's rim. Find the minimum separation resolved by this microscope when an object is illuminated by light with wavelength λ = 0.55 p,m.

Ans.



Let A and B be two points in the field of a microscope which is represented by the lens C D. Let A' ,2T be their image points which are at equal distances from the axis of the lens CD. Then all paths from A to A' are equal and the extreme difference of paths from A to B' is equal to

$$\begin{aligned}
&ADB' - ACB' \\
&= AD + DB' - (AC + CB') \\
&= AD + DB' - BD - DB' \\
&\quad + BC + CB' - AC - CB' \\
&(\text{as } BD + DB' = BC + CB') \\
&= AD - BD + BC - AC \\
&= 2AB\cos(90^\circ - \alpha) = 2AB\sin\alpha
\end{aligned}$$

From the theory of diffraction by circular apertures this distance must be equal to 1.22 λ when B’ coincides with the minimum of the diffraction due to A and A' with the minimum of the diffraction due to B. Thus

$$AB = \frac{1{\cdot}22\,\lambda}{2\sin\alpha} = 0{\cdot}61\frac{\lambda}{\sin\alpha}$$

Here 2α is the angle subtended by the objective of the microscope at the object. Substituting the values

$$AB = \frac{0{\cdot}61 \times 0{\cdot}55}{0{\cdot}24}\,\mu\text{m} = 1{\cdot}40\,\mu\text{m}.$$

Q.150. Find the minimum magnification of a microscope, whose objective's numerical aperture is $\sin\alpha = 0.24$, at which the resolving power of the objective is totally employed if the diameter of the eye's pupil is $d_o = 4.0$ mm.

Ans. Suppose d_{mn} - minimum separation resolved by the microscope ψ = angle subtended at the eye by this object when the object is at the least distance of distinct vision l_0 (= 2 5 cm) .

ψ' minimum angular separation resolved by the eye = $\frac{1{\cdot}22\,\lambda}{d_0}$

From the previous problem

$$d_{min} = \frac{0{\cdot}61\,\lambda}{\sin\alpha}$$

and

$$\psi = \frac{d_{min}}{l_0} = \frac{0{\cdot}61\,\lambda}{l_0 \sin\alpha}$$

Now

$$\Gamma = \text{magnifying power} = \frac{\text{angle subtended at the eye by the image}}{\text{angle subtended at the eye by the object}}$$

when the object is at the least distance of distinct vision

$$\geq \frac{\psi'}{\psi} = 2\left(\frac{l_0}{d_0}\right)\sin\alpha$$

Thus

$$\Gamma_{min} = 2\left(\frac{l_0}{d_0}\right)\sin\alpha = 2 \times \frac{25}{0{\cdot}4} \times 0{\cdot}24 = 30$$

Q.151. A beam of X-rays with wavelength λ falls at a glancing angle 60.0° on a linear chain of scattering centres with period a. Find the angles of incidence corres- ponding to all diffraction maxima if $\lambda = 2a/5$.

Ans. Path difference

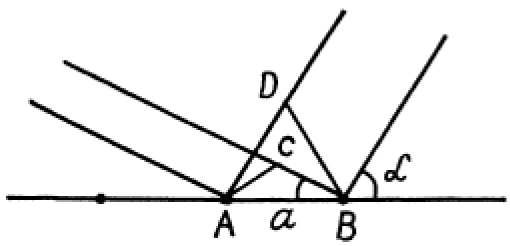
$= BC - AD$

$= a(\cos 60° - \cos \alpha)$

For diffraction maxima

$a(\cos 60° - \cos \alpha) = k\lambda,$

since $\lambda = \frac{2}{5}a$, we get



$$\cos \alpha = \frac{1}{2} - \frac{2}{5}k$$

And we get

$$k = -1, \cos \alpha = \frac{1}{2} + \frac{2}{5} = 0.9, \alpha = 26°$$
$$k = 0, \cos \alpha = \frac{1}{2} = 0.5, \alpha = 60°$$
$$k = 1, \cos \alpha = \frac{1}{2} - \frac{2}{5} = 0.1, \alpha = 84°$$
$$k = 2, \cos \alpha = \frac{1}{2} - \frac{4}{5} = -0.3, \alpha = 107.5°$$
$$k = 3, \cos \alpha = \frac{1}{2} - \frac{6}{5} = -0.7, \alpha = 134.4°$$

Other values o f k are not allowed as they lead to $|\cos \alpha| > 1$.

Q.152. A beam of X-rays with wavelength λ = 40 μm falls normally on a plane rectangular array of scattering centres and produces a system of diffraction maxima (Fig. 5.29) on a plane screen re- moved from the array by a distance l = 10 cm. Find the array periods a and b along the x and y axes if the distances between symmetrically located maxima of second order are equal to Δx = 60 mm (along the x axis) and Δy = 40 mm (along the y axis).

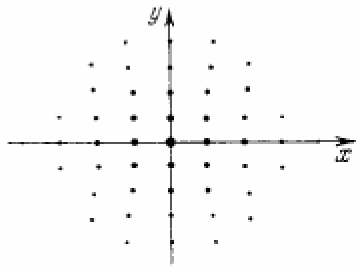
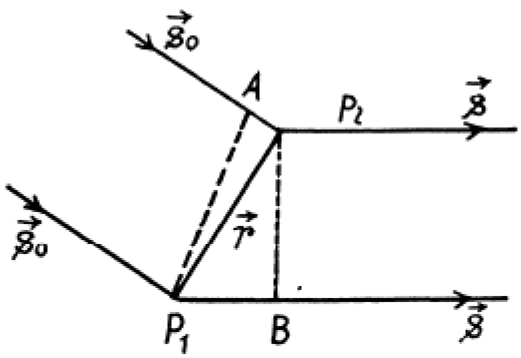


Fig. 5.29.

Ans. We give here a simple derivation of the condition for diffraction maxima, known as Laue equations. It is easy to see form the above figure that the path difference between waves scattered by nearby scattering centres P_1 and P_2 is

$$P_2A - P_1B = \vec{r}\cdot\vec{s_0} - \vec{r}\cdot\vec{s}$$
$$= \vec{r}\cdot(\vec{s_0} - \vec{s}) = \vec{r}\cdot\vec{S}.$$

Here $\vec{r}$ is the radius vector $\overrightarrow{P_1P_2}$. For



maxima this path difference must be an integer multiple of λ, for any two neighbouring atoms. In the present case of two dimensional lattice with X - rays incident normally

$\vec{r}\cdot\vec{s} = 0.$ Taking successively nearest neighbors in the x - & y - directions

$$a\cos\alpha = h\lambda$$
$$b\cos\beta = k\lambda$$

Here $\cos\alpha$ and $\cos\beta$ are the direction cosines of the ray with respect to the x & y axes of the two dimensional crystal.

$$\cos\alpha = \frac{\Delta x}{\sqrt{(\Delta x)^2 + 4l^2}} = \sin\left(\tan^{-1}\frac{\Delta x}{2l}\right) = 0{\cdot}28735$$

so using $h = k = 2$ we get

$$a = \frac{40 \times 2}{\cdot 28735} \text{ pm} = 0{\cdot}278 \text{ nm}$$

Similarly $$\cos\beta = \frac{\Delta y}{\sqrt{(\Delta y)^2 + 4 l^2}} = \sin\left(\tan^{-1}\frac{\Delta y}{2 l}\right) = 0{\cdot}19612$$

$$b = \frac{80}{\cos\beta} \text{ p m} = 0{\cdot}408 \text{ nm}$$

Q.153. A beam of X-rays impinges on a three-dimensional rectangular array whose periods are a, b, and c. The direction of the incident beam coincides with the direction along which the array period is equal to a. Find the directions to the diffraction maxima and the wavelengths at which these maxima will be observed.

Ans. Suppose α, β, and γ are the angles between the direction to the diffraction maximum and the directions of the array along the periods a, b and c respectively (call them x, y, & z axes). Then the value of these angles can be found from the following familiar conditions

$$a(1 - \cos\alpha) = k_1 \lambda$$

$$b\cos\beta = k_2 \lambda \quad \text{and} \quad c\cos\gamma = k_3 \lambda$$

where k_1, k_2, k_3 are whole numbers (+, −, or 0)

(These formulas are, in effect, Laue equations, see any text book on modem physics). Squaring and adding we g et on using $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

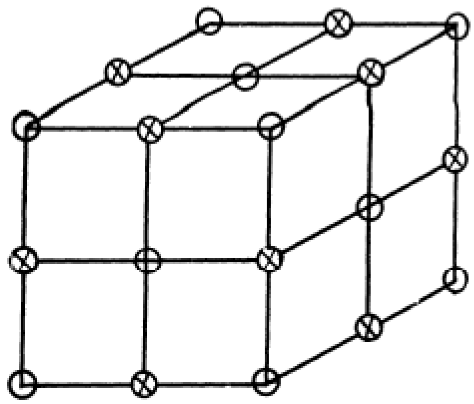
$$2 - 2\cos\alpha = \left[\left(\frac{k_1}{a}\right)^2 + \left(\frac{k_2}{b}\right)^2 + \left(\frac{k_3}{c}\right)^2\right]\lambda^2 = \frac{2 k_1 \lambda}{a}$$

Thus $$\lambda = \frac{2 k_1 / a}{\left[(k_1/a)^2 + (k_2/a)^2 + (k_3/a)^2\right]}$$

Knowing a, b, c and the integer k_1, k_2, k_3 we can find α, β, γ as well as λ.

Q.154. A narrow beam of X-rays impinges on the natural facet of a NaCl single crystal, whose density is p = 2.16 g/cm^3 at a glancing angle θ = 60.0°. The mirror reflection from this facet produces a maximum of second order. Find the wavelength of radiation.

Ans. The unit cell of NaCl is shown below. In an infinite crystal, there are four Na^+ and four Cl^- ions per unit cell. (Each ion on the middle of the edge is shared by four unit cells; each ion on the face centre by two unit cells, the ion in the middle of the cell by one cell only and finally each ion on the comer by eight unit cells.) Thus



$$4\frac{M}{N_A} = \rho \cdot a^3$$

where M = molecular weight of NaCl in gms = 58·5 gms

N_A = Avogadro number = $6{\cdot}023 \times 10^{23}$

Thus $$\frac{1}{2}a = \sqrt{\frac{M}{2N_A\rho}} = 2{\cdot}822\text{ Å}$$

The natural facet of the crystal is one of the faces of the unit cell. The interplanar distance

$$d = \frac{1}{2}a = 2{\cdot}822\text{ Å}$$

Thus $$2d\sin\alpha = 2\lambda$$

So $$\lambda = d\sin\alpha = 2{\cdot}822\text{ Å} \times \frac{\sqrt{3}}{2} = 244\text{ pm.}$$

Q.155. A beam of X-rays with wavelength λ = 174 pm falls on the surface of a single crystal rotating about its axis which is parallel to its surface and perpendicular to the direction of the incident beam. In this case the directions to the maxima of second and third order from the system of planes parallel to the surface of the single crystal form an angle θ = 60° between them. Find the corresponding interplanar distance.

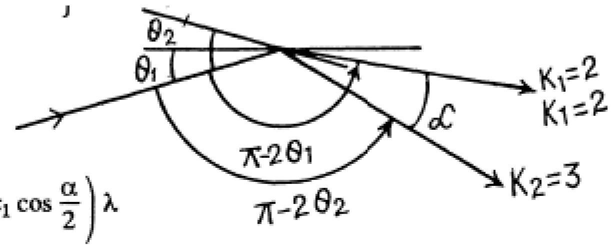
Ans. When the crystal is rotated, the incident monochromatic beam is diffracted from a given crystal plane of interplanar spacing d whenever in the cours V of rotation the value of θ satisfies the Bragg equation.

We have the equations $2d\sin\theta_1 = k_1\lambda$ and $2d\sin\theta_2 = k_2\lambda$

But $\pi - 2\theta_1 = \pi - 2\theta_2 + \alpha$ or $2\theta_1 = 2\theta_2 - \alpha$

so $$\theta_2 = \theta_1 + \frac{\alpha}{2}.$$

Thus $$2d\left\{\sin\theta_1\cos\frac{\alpha}{2} + \cos\theta_1\sin\frac{\alpha}{2}\right\} = k_2\lambda$$



Hence $\quad 2d\sin\frac{\alpha}{2}\cos\theta_1 = \left(k_2 - k_1\cos\frac{\alpha}{2}\right)\lambda$

also $\quad 2d\sin\frac{\alpha}{2}\sin\theta_1 = k_1\lambda\sin\frac{\alpha}{2}$

Squaring and adding $\quad 2d\sin\frac{\alpha}{2} = \left(k_1^2 + k_2^2 - 2k_1k_2\cos\frac{\alpha}{2}\right)^{1/2}\lambda$

Hence
$$d = \frac{\lambda}{2\sin\frac{\alpha}{2}}\left[k_1^2 + k_2^2 - 2k_1k_2\cos\frac{\alpha}{2}\right]^{1/2}$$

Substituting $\quad \alpha = 60°,\ k_1 = 2,\ k_2 = 3,\ \lambda = 174\text{ p m}$

we get $\quad d = 281\text{ p m} = 2{\cdot}81\text{ Å}$

(and not 0·281 p m as given in the book.)
(Lattice parameters are typically in Å's and not in fractions of a pm.)

Q.156. On transmitting a beam of X-rays with wavelength λ = 17.8 pm through a polycrystalline specimen a system of diffraction rings is produced on a screen located at a distance l = 15 cm from the specimen. Determine the radius of the bright ring corresponding to second order of reflection from the system of planes with interplanar distance d = 155 pm.

Ans. In a polycrystalline specimen, microcrystals are oriented at various angles with respect to one another. The microcrystals which are oriented at certain special angles with respect to the incident beam produce diffraction maxima that appear as rings. The radial of these rings are given by

$$r = l\tan 2\alpha$$

where the Bragg's law gives

$$2d\sin\alpha = k\lambda$$

In our case $k = 2,\ d = 155\text{ p m},\ \lambda = 17{\cdot}8\text{ p m}$

so $\alpha = \sin^{-1}\frac{17{\cdot}8}{155} = 6{\cdot}6°$ and $r = 3{\cdot}52$ cm.