Chapter 10

MISCELLANEOUS
THE HODOGRAPH. MOTION ON REVOLVING CURVES. IMPULSIVE TENSIONS OF STRINGS

End of Art 135
EXAMPLES

1. If P be any point on the parabola, S its focus, PK the perpendicular on the directrix, and SY the perpendicular on the tangent at P, then

$$v^2 = 2g \,.\, PK = 2g \,.\, SP = \frac{2g}{a} \,.\, SY^2.$$

Hence SY is perpendicular to, and varies as, the velocity. The hodograph is thus the locus of Y turned through a right angle, *i.e.* it is a vertical straight line, and it is described with uniform velocity, since the velocity in the hodograph is equal to the acceleration in the path.

2. The polar reciprocal of a conic with respect to its focus is a circle, and if the conic is a parabola this circle passes through the focus. Hence, etc., by Art. 135.

3. By Art. 40, $v = \sqrt{\mu} \,.\, CD$. Hence CD is parallel and proportional to the velocity. Hence, etc.

4. $v^2 = 2g\,(a - a\cos\theta) = 2ga\sin^2\frac{\theta}{2}$. Hence, etc.

5. $v = \frac{h}{SY} = \frac{h}{a + a\cos\theta}$. Hence the hodograph is the parabola $r = \frac{\lambda}{1 + \cos\theta}$ turned through a right angle about the centre of force.

6. For the hodograph; $y^2 = \lambda x$, and $\frac{dy}{dt} = V$, so that $y = Vt$ and $x = \frac{V^2}{\lambda} t^2$.

For the path; $\frac{dX}{dt} = kx = \frac{kV^2}{\lambda} t^2$, and $\frac{dY}{dt} = ky = kVt$.

$\therefore\ X = \frac{kV^2}{3\lambda} t^3 + A$, and $Y = \frac{kV}{2} t^2 + B$, so that $(Y - B)^3 = \frac{9k\lambda^2}{8V}(X - A)^2$.

7. With the notation of Art. 100, the figure being inverted, we have if the particle start at a depth $a - b$ below the vertex

$$v^2 = 2g\,[2a\cos^2 QAD - (a - b)] = 2g\,[a\cos 2\theta + b], \text{ where } \theta = \angle QAD.$$

If the particle starts from the highest point, then $b = a$ and $v = 2\sqrt{ga}\cos\theta$, so that the hodograph is a circle.

8. $v = \frac{h}{SY} = \frac{h}{r\sin\alpha} = \frac{h}{a\sin\alpha}e^{-\theta\cot\alpha}$. Hence the hodograph is the equiangular spiral $r = \lambda e^{-\theta\cot\alpha}$ turned through an angle α about the pole.

9. S the pole, P any point on the curve, SY the perpendicular upon the tangent PT. Then, since

$$r^2 = a^2\cos 2\theta,\quad \tan SPT = \frac{rd\theta}{dr} = -\cot 2\theta = \tan\left(\frac{\pi}{2}+2\theta\right),$$

so that $\phi = \angle YSX = 3\theta$. Hence

$$v^2 = \frac{h^2}{SY^2} = \frac{h^2}{r^2\cos^2 2\theta} = \frac{h^2}{a^2}\sec^3\frac{2\phi}{3}.$$

Hence the hodograph is the curve $r^2 = \lambda^2\sec^3\frac{2}{3}\phi$ turned through a right angle, *i.e.* it is the curve

$$r^2 = \lambda^2\sec^3\left(\frac{2}{3}\cdot\overline{\frac{\pi}{2}-\phi}\right) = \lambda^2\sec^3\frac{\pi-2\phi}{3}.$$

10. (x, y) any point on the curve and (ξ, η) the corresponding point on the circle. Then

$$\frac{dx}{dt} = \lambda\xi = \lambda(a + a\cos\omega t),\text{ and } \frac{dy}{dt} = \lambda\eta = \lambda a\sin\omega t.$$

$$\therefore \frac{x}{a} = \lambda t + \frac{\lambda}{\omega}\sin\omega t,\text{ and } \frac{y}{a} = \frac{\lambda}{\omega}(1-\cos\omega t),$$

if the particle is at the origin when $t=0$. Hence, etc.

12. $x = a\cos\theta$, $y = a\sin\theta$, and $z = a\theta\tan\alpha$, so that $z = \frac{a}{\cos\alpha}\theta$.

Then $$X^2 = \lambda v^2\left(\frac{dx}{ds}\right)^2 = \lambda\,.\,2g\sin^2\theta\cos^2\alpha\,(b - a\theta\tan\alpha),$$

$$Y^2 = \lambda v^2\left(\frac{dy}{ds}\right)^2 = \lambda\,.\,2g\cos^2\theta\cos^2\alpha\,(b - a\theta\tan\alpha),$$

and $$Z^2 = \lambda v^2\left(\frac{dz}{ds}\right)^2 = \lambda\,.\,2g\sin^2\alpha\,(b - a\theta\tan\alpha).$$

$$\therefore X^2 + Y^2 = Z^2\cot^2\alpha,\text{ and } Z^2 = 2\lambda g\sin^2\alpha\left(b - a\tan\alpha\tan^{-1}\frac{X}{Y}\right).$$

Hence, etc.

End of Art 140
EXAMPLES

1. The perpendicular OC upon the tube revolves with constant angular velocity ω, so that the acceleration of C is $a\omega^2$ along CO and is zero perpendicular to CO. Hence, if $CP=r$, we have

$$\ddot{r}-r\omega^2=0, \text{ and } \frac{R}{m}=a\omega^2+\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=a\omega^2+2\omega\dot{r}.$$

Hence $r=A\sinh(\omega t+B)$. Also, when $t=0$, $r=0$ and $\dot{r}=-a\omega$.

$$\therefore\ r=-a\sinh\omega t, \text{ and } \frac{R}{m}=-m\omega^2[2\cosh\omega t-1].$$

2. For the motion of the particle with respect to the tube put on an extra force $m\omega^2 a\sin\theta$ outwards from the given diameter.

Then $a\ddot{\theta}=-g\sin\theta+ng\sin\theta\cos\theta$, and $a\dot{\theta}^2=\frac{R}{m}-ng\sin^2\theta-g\cos\theta$.

$$\therefore\ a\dot{\theta}^2=2g\cos\theta+ng\sin^2\theta+2g, \text{ since } \dot{\theta}=0 \text{ when } \theta=\pi.$$

$$\therefore\ t\sqrt{\frac{g}{a}}=\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}\cdot\sec^2\frac{\theta}{2}\,d\theta}{\sqrt{1+(1+n)\tan^2\frac{\theta}{2}}}$$

$$=\frac{1}{\sqrt{1+n}}\log\left[\sqrt{1+n}\tan\frac{\theta}{2}+\sqrt{1+(1+n)\tan^2\frac{\theta}{2}}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\sqrt{1+n}}\log\left[\sqrt{1+n}+\sqrt{2+n}\right].$$

3. As in Ex. 2, $a\dot{\theta}^2=2g\cos\theta+\omega^2 a\sin^2\theta+B$

$$=\omega^2 a\left[2\cos^2\frac{\alpha}{2}\cos\theta+\sin^2\theta-2\cos^2\frac{\alpha}{2}\right], \text{ since } \dot{\theta}=0 \text{ when } \theta=\alpha.$$

Hence
$$\dot{\theta}^2=4\omega^2\sin^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}-\cos^2\frac{\alpha}{2}\right).$$

$$\therefore\ \omega t=\frac{1}{2\sin\frac{\alpha}{2}}\int\frac{\operatorname{cosec}^2\frac{\theta}{2}\,d\theta}{\sqrt{\cot^2\frac{\theta}{2}-\cot^2\frac{\alpha}{2}}}=-\frac{1}{\sin\frac{\alpha}{2}}\cosh^{-1}\left(\frac{\cot\frac{\theta}{2}}{\cot\frac{\alpha}{2}}\right),$$

since $\theta=\alpha$ when $t=0$. Hence, etc.

4. $a\ddot{\theta}=\omega^2 a\sin\theta\cos\theta-g\sin\theta+\frac{\lambda}{a}\left(2a\cos\frac{\theta}{2}-a\right)\sin\frac{\theta}{2}$.

$$\therefore\ a\dot{\theta}^2=\frac{\omega^2 a}{2}(1-\cos 2\theta)+2(\lambda-g)(1-\cos\theta)-4\lambda\left(1-\cos\frac{\theta}{2}\right).\ \ \ldots\ldots(1)$$

This holds until the string becomes unstretched, *i.e.* until

$$\cos\frac{\theta}{2}=\frac{1}{2}, \text{ when } \theta=\frac{2\pi}{3}.$$

The motion is then given by

$$a\dot{\theta}^2=\frac{\omega^2 a}{2}(1-\cos 2\theta)-2g(1-\cos\theta)+A.\ \ldots\ldots\ldots\ldots\ldots(2)$$

The value of $\dot{\theta}$ given by (1) and (2) must be the same when $\theta=\frac{2\pi}{3}$, so that $A=\lambda$. From (2) $\dot{\theta}=0$ when $\theta=\pi$, if $\lambda=4g$. Hence, etc.

5. As before the relative motion is given by

$$a\ddot{\theta}=\omega^2 a(1-\cos\theta)\sin\theta, \text{ so that } \dot{\theta}^2=2\omega^2[1-\cos\theta]-\omega^2\sin^2\theta.$$

$$\therefore\ \omega t=\frac{1}{2}\int_{\pi}^{\pi+\phi}\frac{d\theta}{\sin^2\frac{\theta}{2}}=\left[-\cot\frac{\theta}{2}\right]_{\pi}^{\pi+\phi}=\tan\frac{\phi}{2}.\quad \therefore\ \phi=2\tan^{-1}(\omega t).$$

6. $a\ddot{\theta}=\omega^2 a\sin\theta\cos\theta-g\sin\theta$. Hence there is relative equilibrium when $\theta=0$, or when $\cos\theta=\frac{g}{\omega^2 a}$.

First, let $\omega^2>\frac{g}{a}$. Let $\cos\alpha=\frac{g}{\omega^2 a}$, and put $\theta=\alpha+\psi$, where ψ is small.

Then $\quad \ddot{\psi}=-\psi\left[\frac{g}{a}\cos\alpha-\omega^2\cos 2\alpha\right]=-\psi\,\frac{a^2\omega^4-g^2}{a^2\omega^2}$, etc.

Secondly, if $\omega^2<\frac{g}{a}$, the relative equilibrium is given by $\theta=0$.

If θ is small, then $a\ddot{\theta}=-\theta(g-\omega^2 a)$, etc.

7. $x^2=4ay$. The acceleration to increase x

$$=\omega^2 x.\frac{x}{\sqrt{x^2+4y^2}}-\frac{g.2y}{\sqrt{x^2+4y^2}}=\frac{2ax\left(\omega^2-\frac{g}{2a}\right)}{\sqrt{x^2+4a^2}}.\ \text{Hence, etc.}$$

8. $v^2=3ga+2\int_0^y\omega^2 y\,dy-2g(2a-x)=3ga+\omega^2y^2+2g(x-2a)$

$$=\frac{g}{a}[y^2+2ax-a^2].$$

Hence $v=0$, when

$$a^2\sin^2\theta(1+\cos\theta)^2+2a^2\cos\theta(1+\cos\theta)-a^2=0,$$

i.e. when $0=\cos\theta[4+2\cos\theta-2\cos^2\theta-\cos^3\theta]=\cos\theta(2-\cos^2\theta)(2+\cos\theta)$, the only solution of which is $\cos\theta=0$.

9. $\frac{v^2}{\rho}=R'$, and $\frac{v\,dv}{ds}=0$, *i.e.* $v^2=V^2$. $\quad\therefore\ R'=\frac{V^2}{\rho}$.

Hence, by Art. 136, $R=R'+2m\omega V=2m\omega V+\frac{V^2}{\rho}$.

10. At time t, let x be the stretched length of a portion of the string whose unstretched length was ξ, and T the tension at distance x. Then

$$T-(T+dT)=\omega^2 x.m\,d\xi,\ \text{where}\ \lambda\frac{dx-d\xi}{d\xi}=T.$$

$$\therefore\ \lambda\frac{d^2x}{d\xi^2}=-m\omega^2 x.\quad \therefore\ x=A\sin\left[\sqrt{\frac{m\omega^2}{\lambda}}\,\xi+B\right]=A\sin\left[\frac{\theta\xi}{a}+B\right].$$

Now $x=0$ when $\xi=0$. Also $T=0$, *i.e.* $\frac{dx}{d\xi}=1$, when $\xi=a$.

$\therefore\ B=0$ and $1=\frac{A\theta}{a}\cos\theta$. Hence, when $\xi=a$, $x=A\sin\theta=\frac{a}{\theta}\tan\theta$.

11. Here $\frac{dr}{ds}=\cos\alpha$, and $\ddot{s}=\omega^2 r\cos\alpha$, *i.e.* $\ddot{r}=\omega^2 r\cos^2\alpha$.

$$\dot{r}^2=\omega^2\cos^2\alpha\,(r^2-a^2\cos^2\alpha).$$

For initially $\dot{s}+a\omega\sin\alpha=0$, *i.e.* $\dot{r}+a\omega\sin\alpha\cos\alpha=0$, *i.e.* $\dot{r}=-a\omega\sin\alpha\cos\alpha$, when $r=a$. Hence $\dot{r}=0$ when $r=a\cos\alpha$.

Also $$R'-m\omega^2 r\sin\alpha=\frac{mv^2}{\rho}=\frac{m\sin\alpha\,.\,v^2}{r}=\frac{m\sin\alpha\,.\,\dot{r}^2}{r\cos^2\alpha}.$$

$$\therefore\ R'=m\omega^2\sin\alpha\left(2r-\frac{a^2\cos^2\alpha}{r}\right).$$

Hence $$R=R'+2m\omega v=R'+2m\omega^2\sqrt{r^2-a^2\cos^2\alpha},$$

so that, when $r=a\cos\alpha$, $R=m\omega^2 a\sin\alpha\cos\alpha$,

and, when $r=a$, $R=m\omega^2\sin\alpha\,(2a-a\cos^2\alpha)+2m\omega^2 a\sin\alpha=$ etc.

12. Put $x+iy=Ae^{pt}$, and we obtain

$$p=\alpha-\frac{k}{2}-i(\omega-\beta),\text{ or }-\left(\alpha+\frac{k}{2}-i(\omega+\beta)\right),\text{ where }\alpha,\ \beta\text{ can be found.}$$

Hence if $\alpha-\frac{k}{2}=\gamma$, $\omega-\beta=\delta$, $m=-\left(\alpha+\frac{k}{2}\right)$, and $n=(\omega+\beta)$,

we have $x+iy=(A+Bi)\,e^{\gamma t}(\cos\delta t-i\sin\delta t)+(C+Di)\,e^{mt}(\cos nt-i\sin nt)$,

so that $$x=e^{\gamma t}[A\cos\delta t+B\sin\delta t]+e^{mt}[C\cos nt+D\sin nt],$$

and $$y=e^{\gamma t}[-A\sin\delta t+B\cos\delta t]+e^{mt}[-C\sin nt+D\cos nt].$$

These reduce to the given equations if we are given that, when $t=0$, then $x=1$, $y=0$, $\dot{x}=\gamma$ and $\dot{y}=-\delta$.

13. The tension of the string$=\lambda\frac{r}{d}$, so that the equations of Art. 51 become, on putting $\frac{\lambda}{dm}=\omega^2 n^2$,

$$[D^2+\omega^2(n^2-1)]\,x-2\omega Dy=0,\ \text{ and }\ [D^2+\omega^2(n^2-1)]\,y+2\omega Dx=0.$$

$$\therefore\ [D^2+\omega^2(n^2-1)](x+iy)+2\omega iD\,(x+iy)=0.$$

Putting $x+iy=Ae^{pt}$, we have $p=\alpha i$ or $-\beta i$, where

$$\alpha=(n-1)\,\omega\ \text{ and }\ \beta=(n+1)\,\omega.$$

$$\therefore\ x+iy=(A+Bi)(\cos\alpha t+i\sin\alpha t)+(C+Di)(\cos\beta t-i\sin\beta t).$$

$$\therefore\ x=A\cos\alpha t-B\sin\alpha t+C\cos\beta t+D\sin\beta t,$$

and $$y=A\sin\alpha t+B\cos\alpha t-C\sin\beta t+D\cos\beta t.$$

Now, when $t=0$, $x=a$, $\dot{x}=0$, $y=0$ and $\dot{y}=0$.

$$\therefore\ \frac{A}{\beta}=\frac{C}{\alpha}=\frac{a}{\alpha+\beta},\text{ and }B=D=0.$$

Hence, on putting $\alpha t=\theta$, we have

$$x=\left[a-\frac{a}{2}\left(1-\frac{1}{n}\right)\right]\cos\theta+\frac{a}{2}\left(1-\frac{1}{n}\right)\cos\frac{n+1}{n-1}\theta,$$

and $$y=\left[a-\frac{a}{2}\left(1-\frac{1}{n}\right)\right]\sin\theta-\frac{a}{2}\left(1-\frac{1}{n}\right)\sin\frac{n+1}{n-1}\theta,$$

which are the equations of a hypocycloid, where the radius of the fixed circle$=a$, and that of the rolling circle

$$=\frac{a}{2}\left[1-\frac{1}{n}\right]=\frac{a}{2}\left[1-\omega\sqrt{\frac{dm}{\lambda}}\right].$$

14. Let A be the starting point, P any point on the tube, PM perpendicular to the vertical through A, and MN the perpendicular upon the vertical axis.

Then the acceleration $\omega^2 NP$ is equivalent to $\omega^2 NM$ and $\omega^2 MP$. Hence, if $AP=x$, we have

$$\ddot{x}=\omega^2 . MP \sin\alpha - g\cos\alpha = \omega^2 \sin^2\alpha\left[x-\frac{g\cos\alpha}{\omega^2\sin^2\alpha}\right].$$

$$\therefore\ x-\frac{g\cos\alpha}{\omega^2\sin^2\alpha}=A\cosh(\omega\sin\alpha t)+B\sinh(\omega\sin\alpha t).$$

Now, when $t=0$, $x=0$, and $\dot{x}=0$. $\therefore\ A=-\frac{g\cos\alpha}{\omega^2\sin^2\alpha}$, and $B=0$.

$$\therefore\ x=\frac{g\cos\alpha}{\omega^2\sin^2\alpha}[1-\cosh(\omega\sin\alpha t)]=-\frac{2g\cos\alpha}{\omega^2\sin^2\alpha}\sinh^2(\tfrac{1}{2}\omega\sin\alpha t).$$

End of Art 143

EXAMPLES

1. Equation (4) of Art. 142 gives

$$\frac{d^2T}{d\theta^2}=T,\ \text{so that}\ T=Ae^{\theta}+Be^{-\theta}=\frac{e^{\pi-\theta}-e^{-(\pi-\theta)}}{e^{\pi}-e^{-\pi}}.T_0,$$

since $T=T_0$ when $\theta=0$, and $T=0$ when $\theta=\pi$.

2. Here $\tan\phi=r\frac{d\theta}{dr}=\frac{2}{3}$, so that $\frac{dr}{ds}=\frac{3}{\sqrt{13}}$ and $p=r.\frac{2}{\sqrt{13}}$, and hence $\rho=r\frac{dr}{dp}=\frac{\sqrt{13}}{2}r$. Hence equation (4) of Art. 142 gives $\frac{d^2T}{dr^2}=\frac{4}{9}\frac{T}{r^2}$. Putting $T=r^n$, we have $n=\frac{4}{3}$ or $-\frac{1}{3}$.

$$\therefore\ T=Ar^{\frac{4}{3}}+Br^{-\frac{1}{3}}=Ce^{2\theta}+De^{-\frac{\theta}{2}},$$

where $T_0=C+D$ and $0=Ce^{2\beta}+De^{-\frac{\beta}{2}}$,

so that $\frac{C}{-1}=\frac{D}{e^{\frac{5\beta}{2}}}=\frac{T_0}{e^{\frac{5\beta}{2}}-1}$. Hence, etc.

3. Here $\tan\phi=\frac{3}{8}$, $\frac{dr}{ds}=\frac{8}{\sqrt{73}}$, $p=\frac{3r}{\sqrt{73}}$ and $\rho=\frac{\sqrt{73}}{3}r$. Hence we have

$$\frac{d^2T}{dr^2}.\frac{64}{73}=\frac{T}{r^2}.\frac{9}{73}.$$

Putting $T=r^n$, we have $n=\frac{9}{8}$ or $-\frac{1}{8}$, so that $T=Ar^{\frac{9}{8}}+Br^{-\frac{1}{8}}$.

But $T=2$ when $r=1$, and $T=1$ when $r=256$, so that $A=0$ and $B=2$. Hence when $r=81$, $T=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$.

4. Let T be the blow, per unit of length, on the upper sphere, so that

$$mV = 2\pi r \,.\, T\cos\alpha = \frac{2\pi r\sqrt{R^2-r^2}}{R}\,.\,T.$$

Also, if T_1 is the impulsive tension in the wire, then

$$T_1\,.\,2\sin\theta = T\sin\alpha\,.\,2r\theta, \text{ where } \theta \text{ is very small.}$$

$$\therefore\ T_1 = r\sin\alpha T = \frac{r^2}{R}\,.\,T = \frac{mVr}{2\pi\sqrt{R^2-r^2}}.$$

5. Let AOB be the horizontal diameter, B being the point at which the vertical part of the string is tangential, and let $f\,(=a\ddot{\psi})$ be the acceleration of the string at time t.

Then, θ being the angular distance of any point of the string in the tube from A, we have

$$ma\delta\theta\,.\,f = (T+dT)\cos\delta\theta - T - mga\,\delta\theta\cos\theta = \delta T - mga\delta\theta\cos\theta.$$

$$\therefore\ maf(\theta-\psi) = T - mga(\sin\theta - \sin\psi), \text{ since } T=0 \text{ when } \theta=\psi.$$

Hence, if T_1 is the tension at B,

$$maf(\pi-\psi) = T_1 + mga\sin\psi.$$

But, for the vertical portion of the string, we have

$$ma\psi\,.\,f = -T_1 + mga\psi.$$

Hence $\quad mf\pi = mg(\psi+\sin\psi)$, *i.e.* $a\pi\ddot{\psi} = g(\psi+\sin\psi)$.

Also $\quad \dfrac{T}{mga} = \sin\theta - \sin\psi + (\theta-\psi)\,.\,\dfrac{\psi+\sin\psi}{\pi}.$

The maximum value of T is given by $\dfrac{dT}{d\theta}=0$, and then $\cos\theta = -\dfrac{\psi+\sin\psi}{\pi}$.

6. At any moment let β and γ be the vectorial angles of the ends of the chain. For any element of the chain, if f be its acceleration, we have

$$m\delta s\,.\,f = (T+\delta T)\cos\delta\psi - T + m\mu r\,.\,\delta s\cos\alpha = \delta T + m\mu s\,.\,\delta s\cos^2\alpha.$$

$$\therefore\ mfs = T + \frac{m\mu}{2}s^2\cos^2\alpha + A\ .\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Since the tension is zero at each end, this gives

$$f\,.\,l = \frac{\mu}{2}[s_\gamma{}^2 - s_\beta{}^2]\cos^2\alpha, \text{ i.e. } f = \frac{\mu}{2}(s_\gamma+s_\beta)\cos^2\alpha.$$

The distance of a particle at the middle point of the chain from the pole $= \dfrac{s_\gamma+s_\beta}{2}\,.\cos\alpha$, and hence its acceleration

$$= \mu\left[\frac{s_\gamma+s_\beta}{2}\cos\alpha\right]\cos\alpha = \frac{\mu}{2}(s_\gamma+s_\beta)\cos^2\alpha = f.$$

Also (1) gives

$$T + \frac{m\mu}{2}(s^2-s_\beta{}^2)\cos^2\alpha = mf(s-s_\beta) = \frac{m\mu}{2}(s_\gamma+s_\beta)(s-s_\beta)\cos^2\alpha.$$

$$\therefore\ T = \frac{m\mu}{2}\cos^2\alpha\,(s-s_\beta)[s_\beta+s_\gamma-(s+s_\beta)] = \frac{m\mu}{2}\cos^2\alpha\,(s-s_\beta)(s_\gamma-s)$$

$$= \frac{m\mu}{2}\,.\,x(l-x)\cos^2\alpha.$$