Relations and Functions

- **Cartesian product of two sets:** Two non-empty sets P and Q are given. The Cartesian product P × Q is the set of all ordered pairs of elements from P and Q, i.e.,

  P × Q = {(p, q) : p ∈ P and q ∈ **Q}**
  Example: If P = {x, y} and Q = {-1, 1, 0}, then P × Q = {(x, -1), (x, 1), (x, 0), (y, -1), (y, 1), (y, 0)}
  If either P or Q is a null set, then P × Q will also be a null set, i.e., P × Q = φ.
  In general, if A is any set, then A × $\varphi = \varphi$.

- **Property of Cartesian product of two sets:**
  - If n(A) = p, n(B) = q, then n(A × B) = pq.
  - If A and B are non-empty sets and either A or B is an infinite set, then so is the case with A × B
  - .A × A × A = {(a, b, c) : a, b, c ∈ A}. Here, (a, b, c) is called an ordered triplet.
  - A × (B ∩ C) = (A × B) ∩ (A × C)
  - A × (B ∪ C) = (A × B) ∪ (A × C)

- Two ordered pairs are equal if and only if the corresponding first elements are equal and the second elements are also equal. In other words, if (*a*, *b*) = (*x*, *y*), then *a* = *x* and *b* = *y*.

  Example: Show that there does not exist *x*, *y* ∈ R if (*x* – *y* + 1, 4*x* – 2*y* – 6) = (*y* – *x* – 4, 7*x* – 5*y* – 2).

  Solution: It is given that (*x* – *y* + 1, 4*x* – 2*y* – 6) = (*y* – *x* – 4, 7*x* – 5*y* – 2).
  ⇒ *x* – *y* + 1 = *y* – *x* – 4 and 4*x* – 2*y* – 6 = 7*x* – 5*y* – 2
  ⇒ 2*x* – 2*y* + 5 = 0 … (1)
  And – 3*x* + 3*y* - 4 = 0 … (2)
  Now,
  $\frac{2}{-3} = -\frac{2}{3}, \frac{-2}{3} = -\frac{2}{3}$ and $\frac{5}{-4} = -\frac{5}{4}$
  Since $\frac{2}{-3} = \frac{-2}{3} \neq \frac{5}{-4}$, equations (1) and (2) have no solutions. This shows that there does not exist *x*, *y* ∈ R if (*x* – *y* + 1, 4*x* – 2*y* – 6) = (*y* - *x* – 4, 7*x* – 5*y* – 2).
  In general, for any two sets A and B, A × B ≠ B × A.

- **Relation:** A relation *R* from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element *x* and the second element *y* of the ordered pairs (*x*, *y*) in A × B.
- The image of an element *x* under a relation *R* is *y*, where (*x*, *y*) ∈ R
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation *R*. Range ⊆Co-domain

  Example: In the relation X from **W** to **R**, given by X = {(*x*, *y*): *y* = 2*x* + 1; *x* ∈ W, *y* ∈ R}, we obtain X = {(0, 1), (1, 3), (2, 5), (3, 7) ...}. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 ...}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 ...}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

  If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

- A relation *R* from a set *A* to a set *B* is said to be a **function** if for every *a* in *A*, there is a unique *b* in *B* such that $(a, b) \in R$.
- If *R* is a function from *A* to *B* and $(a, b) \in R$, then *b* is called the **image** of *a* under the relation *R* and *a* is called the **preimage** of *b* under *R*.
- For a function *R* from set *A* to set *B*, set *A* is the **domain** of the function; the images of the elements in set *A* or the second elements in the ordered pairs form the **range**,while the whole of set *B* is the **codomain** of the function.

  For example, in relation $f = \{(-1,3)(0,2),(1,3),(2,6),(3,11)\}$ since each element in *A* has a unique image, therefore *f* is a function.

  Each image in *B* is 2 more than the square of pre-image.

  Hence, the formula for *f* is $f(x) = x^2 + 2$ Or $f : x \to x^2 + 2$
  Domain = {−1, 0, 1, 2, 3}
  Co-domain = {2, 3, 6, 11, 13}
  Range = {2, 6, 3, 11}

- **Real-valued Function:** A function having either R (real numbers) or one of its subsets as its range is called a real-valued function. Further, if its domain is also either R or *a* subset of R, it is called a real function.

  Types of functions:

  - **Identity function:** The function *f*: R → R defined by $y = f(x) = x$, for each $x \in$ R, is called the identity function.

    Here, R is the domain and range of *f*.

- **Constant function:** The function f: R → R defined by $y = f(x) = x$, for each $x \in$ R, where c is a constant, is a constant function.

  Here, the domain of f is R and its range is $\{c\}$.

- **Polynomial function:** A function f: R → R is said to be a polynomial function if for each $x \in$ R, $y = f(x) = a^0 + a_1x +$ ___$+ a_n x^n$a where n is a non-negative integer and $a_0, a_1, \ldots, a_n \in$ R .

- **Rational function:** The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$, are called rational functions.

- **Modulus function:** The function f: R → R^+ defined by $f(x) = |x|$, for each $x \in$ R, is called the modulus function.

  In other words, $$f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

- **Signum function:** The function f: R → R defined by

  $$f(x) = \begin{Bmatrix} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{Bmatrix}$$

  is called the signum function. Its domain is R and its range is the set {–1, 0, 1}.

- **Greatest Integer function**: The function f: R → R defined by $f(x) = [x]$, $x \in$ R, assuming the value of the greatest integer less than or equal to x, is called the greatest integer function.

  Example: [–2.7] = –3, [2.7] = 2, [2] = 2

- **Linear function:** The function f defined by $f(x) = mx + c$, for $x \in$ R, where m and c are constants, is called the linear function. Here, R is the domain and range of f.

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define

- Addition of Functions
  $(f + g)$: X → R by $(f + g)(x) = f(x) + g(x)$, $x \in$ X
- Subtraction of Functions
  $(f - g)$: X → R by $(f - g)(x) = f(x) - g(x)$, $x \in$ X

  Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then
  $(f + g)(x) = f(x) + g(x)$
  $= (2x - 3) + (x^2 + 3x + 2)$
  $= x^2 + 5x - 1$
  $(f - g) = f(x) - g(x)$
  $= (2x - 3) - (x^2 + 3x + 2)$

$= -x^2 - x - 5$

- **Multiplication of real functions:** For functions f: X → R and g: X → R, we define Multiplication of two real functions
  (fg): X → R by $(fg)(x) = f(x).\ g(x)$ $x \in X$

  - Multiplication of a function by a scalar
    (af): X → R by $(a\,f)\ (x) = af\ (x)$ $x \in X$ and a is a real number

    Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then
    $(fg)\ (x) = f(x) \times g(x)$
    $= (2x - 3) \times (x^2 + 3x + 2)$
    $= 2x^3 + 3x^2 - 5x - 6$
    $(\ 2f\ )(x) = 2.f(x)$
    $= 2\times(2x - 3)$
    $= 4x - 6$

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    $(f + g)$: X → R by $(f + g)\ (x) = f(x) + g(x)$, $x \in X$
  - Subtraction of Functions
    $(f - g)$: X → R by $(f - g)\ (x) = f(x) - g(x)$, $x \in X$

    Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then
    $(f + g)\ (x) = f(x) + g(x)$
    $= (2x - 3) + (x^2 + 3x + 2)$
    $= x^2 + 5x - 1$
    $(f - g) = f(x) - g(x)$
    $= (2x - 3) - (x^2 + 3x + 2)$
    $= -x^2 - x - 5$