Exercise 14.8

Answer 1E.

We need to find the extreme values of "f" subject is a constraint $g(x, y) = 8$, or we need the extreme values of "f" when the point (x, y) is restricted to lie on the level curve $g(x, y) = 8$

The figure shows the curve together with several level curves of "f". To maximize $f(x, y)$ subject to $g(x, y) = 8$, we find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = 8$. This happens when these curves just touch each other, that is when they have a common tangent line. Therefore from the figure we find that the maximum value of $f = \boxed{59}$.

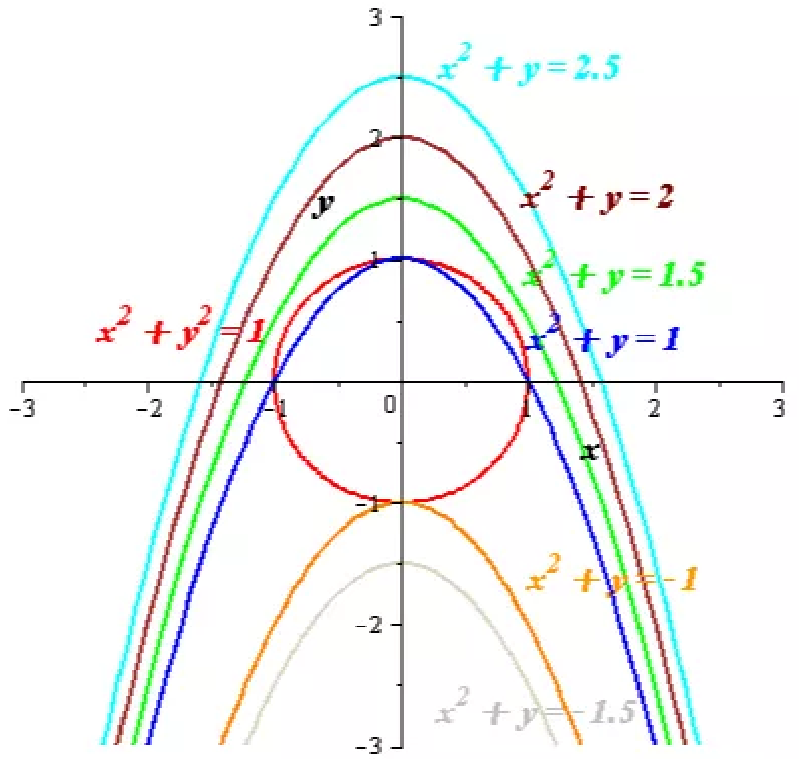
Similarly to minimize $f(x, y)$ subject to $g(x, y) = 8$, we find the smallest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y)$, which happens when these curves just touch each other that is they have a common tangent. From the figure we find the minimum value of $f = \boxed{30}$

Answer 2E.

Method of Lagrange Multipliers:

Let $f(x, y, z)$ be the function subject to constraint $g(x, y, z) = k$. Assume that the extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$. Find all values of x, y, z and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$. Now, evaluate f at all the points that come from above step. The largest of these values is maximum value of f and the smallest is the minimum value of f.

Use maple to draw the graph of $x^2 + y^2 = 1$ and curves $x^2 + y = c$ for different values of c.



(a)

From the graph, observe that when c = 1 and -1, the curves $x^2 + y = c$ just touch the circle.

When c = 1, it is the largest value of c such that the level curve $x^2 + y = c$ intersects $x^2 + y^2 = 1$. This gives the maximum value of function $f(x,y) = x^2 + y$

And when c = -1, it gives the minimum value of the function $f(x,y) = x^2 + y$

(b)

The objective is to find maximum and minimum values of f by Lagrange multiplier method.

Given $f(x,y) = x^2 + y$

And the constraint is; $g(x,y): x^2 + y^2 = 1$

From Lagrange's method of multiplier find x, y and λ such that $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$

And $g(x,y) = k$

Now, $f_x = \lambda g_x$, $f_y = \lambda g_y$

And $g(x,y) = k$

So, $2x = \lambda 2x$, $1 = \lambda 2y$

And $x^2 + y^2 = 1$

From first equation $\lambda = 1$ or $x = 0$

And then $\frac{1}{2}$ or $y = \pm 1$

Therefore;

$$x^2 = 1 - y^2$$
$$= 1 - \frac{1}{4}$$
$$x^2 = \frac{3}{4}$$
$$x = \pm \frac{\sqrt{3}}{2}$$

Therefore, f has possible extreme values at points $(0,1)$, $(0,-1)$, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$

Evaluate f on these points;

$$f(0,1) = 1$$
$$f(0,-1) = -1$$
$$f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2}$$
$$= \frac{5}{4}$$
$$f\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2}$$
$$= \frac{5}{4}$$

Therefore maximum value of f is $\boxed{\frac{5}{4}}$ at $\left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

And minimum value of f is -1 at $(0,-1)$.

Answer 3E.

We wish to minimize and maximize

$f(x,y) = x^2 + y^2$

subject to the constraint

$g(x,y) = xy = 1$

Use Lagrange multipliers as follows. Calculate $\nabla f = \lambda \nabla g$. Along with the constraint equation $g(x,y) = 1$, this will provide a system of three equations in three variables.

The solution (x,y) to these equations that gives the largest value of f will be giving the maximum value of f, the solution that gives the smallest value of f will be giving the minimum value of f.

Find the equations resulting from $\nabla f = \lambda \nabla g$. When taking partial derivatives, hold one variable constant and differentiate with respect to the other, as follows:

$$f_x = \lambda g_x$$
$$2x = \lambda y$$
$$f_y = \lambda g_y \qquad \ldots\ldots (1)$$
$$2y = \lambda x$$

Solve the system of equations formed by the equations in (1), $2x = \lambda y$ and $2y = \lambda x$, and the constraint equation $xy = 1$. First use the first equation in (1) to get an expression for λ:

$$\lambda = \frac{2x}{y}$$

Plug this into the second equation in (1):

$$2y = \lambda x$$
$$\Rightarrow 2y = \left(\frac{2x}{y}\right)x \qquad \ldots\ldots (2)$$
$$\Rightarrow 2y^2 = 2x^2$$
$$\Rightarrow y^2 = x^2$$

Now solve the constraint equation $xy = 1$ for y and substitute it in to (2):

$$y = \frac{1}{x}$$
$$\left(\frac{1}{x}\right)^2 = x^2$$
$$x^4 = 1$$
$$x = \pm 1$$

Back-substituting into $y = \frac{1}{x}$ gives the points (1,1) and (-1,-1) as the (x, y) solutions to the system.

We plug the points (1,1) and (-1,-1) into the function *f*, the largest result should be the maximum value of *f* and the smallest result the minimum value.

$$\begin{aligned} f(1,1) &= 1^2 + 1^2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} f(-1,-1) &= (-1)^2 + (-1)^2 \\ &= 2 \end{aligned}$$

Since *f* has the same value at both, it must have either a maximal value or a minimal value at both points. We check to see which it is by plugging in another arbitrary (*x*,*y*) value that satisfies the constraint equation $xy = 1$ and seeing whether the function value is greater or less than 2. Choose $\left(2, \frac{1}{2}\right)$:

$$\begin{aligned} f\left(2, \frac{1}{2}\right) &= 2^2 + \left(\frac{1}{2}\right)^2 \\ &= 4 + \frac{1}{4} \\ &= \frac{17}{4} \end{aligned}$$

which is greater than 2. So 2 must be the minimal value for *f*.

The function *f* has a minimum equal to 2 at (1,1) and (-1,-1), and has no maximum.

Answer 4E.

We have $f(x, y) = 3x + y$ and $x^2 + y^2 = 10$. Let $g(x, y) = x^2 + y^2 - 10$.

Find ∇f and ∇g.

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \langle 3, 1 \rangle \end{aligned}$$

$$\begin{aligned} \nabla g &= \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle \\ &= \langle 2x, 2y \rangle \end{aligned}$$

Let $\nabla f = \lambda \nabla g$.

$3 = 2\lambda x$ ....(1)

$1 = 2\lambda y$ ....(2)

$x^2 + y^2 = 10$ ....(3)

Now, from (1), we get $x = \frac{3}{2\lambda}$ and from (2), we get $y = \frac{1}{2\lambda}$.

Replace x with $\frac{3}{2\lambda}$ and y with $\frac{1}{2\lambda}$ in $x^2 + y^2 = 10$ and solve for λ.

$$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10$$

$$4\lambda^2 = 1$$

$$\lambda^2 = \frac{1}{4}$$

$$\lambda = \pm\frac{1}{2}$$

Then, we get (3, 1) and (−3, −1).

The maximum value of f is $f(3,1) = 10$ and the minimum value of f is $f(-3,-1) = -10$.

Answer 5E.

Given $f(x,y) = y^2 - x^2$

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0, z_0) on the smooth constraint curve $g(x,y,z) = k$.

If $\nabla g(x, y, z) \neq 0$, then there is a real number λ such that

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

Let $g(x, y) = \frac{1}{4}x^2 + y^2 - 1$.

We know that gradient of f is given by $\nabla f = \langle f_x, f_y \rangle$.

Find $\nabla f(x, y)$ and $\nabla g(x, y)$.

$$\nabla f = \langle -2x, 2y \rangle$$

$$\nabla g = \left\langle \frac{x}{2}, 2y \right\rangle$$

Then, $\nabla f = \lambda \nabla g$ or $\langle -2x, 2y \rangle = \lambda \left\langle \frac{x}{2}, 2y \right\rangle$.

On equating the like terms, we get $-2x = \frac{x\lambda}{2}$ or $x(\lambda+4) = 0$ and $2y = 2y\lambda$ or $2y(1-\lambda) = 0$. From $x(\lambda+4) = 0$, we can say $x=0$ or $\lambda = 4$.

Replace x with 0 in $\frac{1}{4}x^2 + y^2 = 1$ and solve for y.

$$\frac{1}{4}(0^2) + y^2 = 1$$

$$y^2 = 1$$

$$y = \pm 1$$

Now, we also have $2y(1-\lambda) = 0$. This means that $y=0$ or $\lambda = 1$.

On substituting 0 for y in $\frac{1}{4}x^2 + y^2 = 1$, we get $x = \pm 2$.

Thus, f has possible extreme values at (0, 1), (0, −1), (2, 0), and (−2, 0).

Now, we also have $2y(1-\lambda) = 0$. This means that $y=0$ or $\lambda = 1$.

On substituting 0 for y in $\frac{1}{4}x^2 + y^2 = 1$, we get $x = \pm 2$.

Thus, f has possible extreme values at (0, 1), (0, −1), (2, 0), and (−2, 0).

Answer 6E.

Given $f(x,y) = e^{xy}$, $x^3 + y^3 = 16 = g(x,y)$

Using Lagrange's multipliers, we solve the equations

$\nabla f = \lambda \nabla g$, $g(x,y) = 16$ which can be written as

$f_x = \lambda g_x$; $f_y = \lambda g_y$; $g(x,y) = 16$

$$\Rightarrow e^{xy} \cdot y = \lambda(3x^2) \text{ and } e^{xy} \cdot x = \lambda(3y^2) \text{ and } x^3 + y^3 = 16$$

(1) (2) (3)

$$\frac{(1)}{(2)} \Rightarrow \frac{e^{xy} y}{e^{xy} \cdot x} = \frac{3\lambda x^2}{3\lambda y^2} \Rightarrow y^3 = x^3$$

From (3):

$$\Rightarrow x^3 + x^3 = 16 \Rightarrow 2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

If $x = 2$ then $y = 2 \Rightarrow (2,2)$

$f(2,2) = e^4$ It has no minimum value

The maximum value is e^4.

Answer 7E.

Given $f(x,y,z) = 2x+2y+z, x^2+y^2+z^2=9$

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0, z_0) on the smooth constraint curve $g(x,y,z)=k$.

If $\nabla g(x,y,z) \neq 0$, then there is a real number λ such that

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

Let $g(x,y,z) = x^2+y^2+z^2-9$.

We know that the gradient of f is given by $\nabla f = \langle f_x, f_y, f_z \rangle$.

Find $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$.

$\nabla f = \langle 2, 2, 1 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

Then, $\nabla f = \lambda \nabla g$ or $\langle 2, 2, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$.

On equating the like terms, we get $x = \dfrac{1}{\lambda}, y = \dfrac{1}{\lambda}$, and $z = \dfrac{1}{2\lambda}$.

Substitute the known values in $x^2+y^2+z^2 = 9$ and solve for λ.

$$\frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 9$$
$$\frac{2}{\lambda^2} + \frac{1}{4\lambda^2} = 9$$
$$9\lambda^2 = \frac{9}{4}$$
$$\lambda = \pm\frac{1}{2}$$

On replacing $\dfrac{1}{2}$ for λ and $-\dfrac{1}{2}$ for λ in the equations for x, y, and z we get $(2,2,1)$ and $(-2,-2,-1)$.

Thus, f has possible extreme values at $(2,2,1)$ and $(-2,-2,-1)$.

Find $f(2,2,1)$ and $f(-2,-2,-1)$.

$$f(2,2,1) = 2(2)+2(2)+1$$
$$= 9$$

$$f(-2,-2,-1) = 2(-2)+2(-2)-1$$
$$= -9$$

Thus, we note that the maximum value of is $f(2,2,1)=9$ and the minimum value is $f(-2,-2,-1)=-9$

Answer 8E.

Use Lagrange multipliers to find the maximum and minimum values of the function

$f(x,y,z)=x^2+y^2+z^2$, subject to the constraint $g(x,y,z)=x+y+z-12$

The method of Lagrange multipliers:

To find the extreme values of $f(x,y)$ subject to the constraint $g(x,y)=k$

Then, solve the following equations for x, y, λ

$\nabla f = \lambda \nabla g$, and $g(x,y)=k$

Now,

$$f(x,y,z)=x^2+y^2+z^2$$

$$\nabla f=\langle 2x,2y,2z\rangle$$

And,

$$g(x,y,z)=x+y+z-12$$

$$\nabla g=\langle 1, 1, 1\rangle$$

Therefore,

$$\nabla f=\lambda\nabla g$$

$$\langle 2x, 2y, 2x\rangle=\lambda\langle 1, 1, 1\rangle$$

$$x=\frac{\lambda}{2}$$

$$y=\frac{\lambda}{2}$$

$$z=\frac{\lambda}{2}$$

Substitute the known values in $x + y + z = 12$ and solve for λ.

$$\frac{\lambda}{2}+\frac{\lambda}{2}+\frac{\lambda}{2}=12$$
$$\frac{3\lambda}{2}=12$$
$$\lambda=8$$

Then, $x=y=z=4$

Then the value of f at the point $(4,4,4)$ is,

$$4^2+4^2+4^2=48$$

Take the arbitrary points $(-12,0,24),(0,0,12)$ on the plane $g(x,y,z)=x+y+z-12$

At these points the values of f are,

$$(-12)^2+0^2+(24)^2=144+576$$
$$=720$$
$$0^2+0^2+(12)^2=144$$

These values exceeds the value of f at the point $(4,4,4)$

Thus, it is clear that any point satisfies $g(x,y,z)=x+y+z-12$ will gives greater value for f, when compared with the value of f at the point $(4,4,4)$

So the minimum value of f at the point $(4,4,4)$ is $\boxed{48}$

And, there is no maximum for f.

Answer 9E.

$$f(x,y,z)=xyz$$
$$g(x,y,z)=x^2+2y^2+3z^2=6$$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla}f(x,y,z)=\lambda\vec{\nabla}g(x,y,z)$$

And $g(x,y,z)=6$

i.e.
$$f_x=\lambda g_x$$
$$f_y=\lambda g_y$$
$$f_z=\lambda g_z$$
$$g(x,y,z)=6$$

i.e. $yz = \lambda 2x$ --------------- (1)

$xz = \lambda 4y$ --------------- (2)

$xy = \lambda 6z$ --------------- (3)

$x^2 + 2y^2 + 3z^2 = 6$ --------------- (4)

From equation (1) $xyz = 2\lambda x^2$

From (2) $xyz = 4\lambda y^2$

From (3) $xyz = 6\lambda z^2$

i.e. $2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2$

i.e. $x^2 = 2y^2 = 3z^2$ $(\lambda \neq 0)$

(Because if $\lambda = 0$ then $x = y = z = 0$ which is not possible because of (4))

Using this in equation (4) we find

$$x = \pm\sqrt{2},\ y = \pm 1,\ z = \pm\sqrt{\frac{2}{3}}$$

Then the possible extreme points of "f" are

$$\left(\sqrt{2},1,\sqrt{\frac{2}{3}}\right), \left(-\sqrt{2},1,\sqrt{\frac{2}{3}}\right), \left(\sqrt{2},-1,\sqrt{\frac{2}{3}}\right), \left(\sqrt{2},1,-\sqrt{\frac{2}{3}}\right)$$

$$\left(-\sqrt{2},-1,\sqrt{\frac{2}{3}}\right), \left(-\sqrt{2},1,-\sqrt{\frac{2}{3}}\right), \left(\sqrt{2},-1,-\sqrt{\frac{2}{3}}\right), \left(-\sqrt{2},-1,-\sqrt{\frac{2}{3}}\right)$$

Then evaluating f at these points

$$f\left(\sqrt{2},1,\sqrt{\frac{2}{3}}\right) = \frac{2}{\sqrt{3}}$$

$$f\left(-\sqrt{2},1,\sqrt{\frac{2}{3}}\right) = \frac{-2}{\sqrt{3}}$$

$$f\left(\sqrt{2},-1,\sqrt{\frac{2}{3}}\right) = \frac{-2}{\sqrt{3}}$$

$$f\left(\sqrt{2},1,-\sqrt{\frac{2}{3}}\right) = \frac{-2}{\sqrt{3}} \quad \text{and}$$

$$f\left(-\sqrt{2},-1,\sqrt{\frac{2}{3}}\right)=\frac{2}{\sqrt{3}}$$

$$f\left(-\sqrt{2},1,-\sqrt{\frac{2}{3}}\right)=\frac{2}{\sqrt{3}}$$

$$f\left(\sqrt{2},-1,-\sqrt{\frac{2}{3}}\right)=\frac{2}{\sqrt{3}}$$

$$f\left(-\sqrt{2},-1,-\sqrt{\frac{2}{3}}\right)=\frac{-2}{\sqrt{3}}$$

Hence the maximum value of "f" is $\boxed{\frac{2}{\sqrt{3}}}$

And minimum value of "f" is $\boxed{\frac{-2}{\sqrt{3}}}$

Answer 10E.

Consider the function $f(x,y,z)=x^2y^2z^2$ and the constraint $x^2+y^2+z^2=1$.

The objective is to find maximum and minimum values of the function using Lagrange multipliers method.

Rewrite the constraint as,

$$\begin{aligned} g(x,y,z) &= x^2+y^2+z^2 \\ &= 1 \end{aligned}$$

The function can be written as,

$$f(x,y,z)=x^2y^2z^2$$

Take the partial derivative with respect to 'x'.

$$f_x(x,y,z)=2xy^2z^2$$

Take the partial derivative with respect to 'y'.

$$f_y(x,y,z)=2x^2yz^2$$

Take the partial derivative with respect to 'z'.

$$f_z(x,y,z)=2x^2y^2z$$

The constraint can be written as,

$$g(x,y,z) = x^2 + y^2 + z^2 - 1$$

Take the partial derivative with respect to ' x '.

$$g_x(x,y,z) = 2x$$

Take the partial derivative with respect to ' y '.

$$g_y(x,y,z) = 2y$$

Take the partial derivative with respect to ' z '.

$$g_z(x,y,z) = 2z$$

From Lagrange multipliers method,

$$f_x = \lambda g_x \quad \text{......(1)}$$
$$f_y = \lambda g_y \quad \text{......(2)}$$
$$f_z = \lambda g_z \quad \text{......(3)}$$
$$x^2 + y^2 + z^2 = 1 \quad \text{......(4)}$$

Substitute the partial derivatives in the above method, then the equations becomes,

$$2xy^2z^2 = \lambda 2x \quad \text{......(5)}$$
$$2x^2yz^2 = \lambda 2y \quad \text{......(6)}$$
$$2x^2y^2z = \lambda 2z \quad \text{......(7)}$$

Multiply equation (5) by x, equation (6) by y and equation (7) by z,

$$2x^2y^2z^2 = \lambda 2x^2$$
$$2x^2y^2z^2 = \lambda 2y^2$$
$$2x^2y^2z^2 = \lambda 2z^2$$

Left hand sides of all three equations are equal.

$$2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$$

The left hand side of all three equations are equal i.e. either $\lambda = 0$ or $x^2 = y^2 = z^2$.

If $\lambda = 0$ then the equations becomes,

$$2x^2y^2z^2 = \lambda 2x^2$$
$$x^2y^2z^2 = 0$$
$$x = 0, \text{ or } y = 0, \text{ or } z = 0$$

If $x^2 = y^2 = z^2$ then the equation (4) becomes,

$$x^2 + y^2 + z^2 = 1$$
$$x^2 + x^2 + x^2 = 1$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{3}$$
$$x = \pm\sqrt{\frac{1}{3}} \quad \left(\text{Since, } y = z = \pm\sqrt{\frac{1}{3}}\right)$$

Therefore, the possible points are $\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$ and $(0,0,0)$ i.e. total 9 points.

The possible points can be written as,

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$
$$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

And $(0,0,0)$.

The value of the function at $\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$ is,

$$f\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right) = x^2y^2z^2$$
$$= \left(\pm\frac{1}{\sqrt{3}}\right)^2 \cdot \left(\pm\frac{1}{\sqrt{3}}\right)^2 \cdot \left(\pm\frac{1}{\sqrt{3}}\right)^2$$
$$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$$
$$= \boxed{\frac{1}{27}}$$

The value of the function at $(0,0,0)$ is,

$$f(0,0,0) = x^2y^2z^2$$
$$= (0)^2(0)^2(0)^2$$
$$= \boxed{0}$$

Hence, the maximum value is $\boxed{\frac{1}{27}}$ and the minimum value is $\boxed{0}$.

Answer 11E.

Consider the function $f(x,y,z) = x^2 + y^2 + z^2$ with constraint $g(x,y,z) = x^4 + y^4 + z^4 = 1$.

Find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$.

Find all values of $x, y, z,$ and λ:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = k$$

Use the method of Lagrange's multipliers, this gives the equations:

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$
$$g(x,y,z) = k$$

Therefore, the equations are as follows:

$$2x = \lambda(4x^3)$$
$$2y = \lambda(4y^3)$$
$$2z = \lambda(4z^3)$$
$$x^4 + y^4 + z^4 = 1$$

By simplifying,

$x = 2\lambda x^3$ ...... (1)

$y = 2\lambda y^3$ ...... (2)

$z = 2\lambda z^3$ ...... (3)

$x^4 + y^4 + z^4 = 1$ ...... (4)

If $x=0$, then y and $z \neq 0$ $(\lambda \neq 0)$.

Then (2) and (3) yield $y^2 = z^2 = \frac{1}{2\lambda}$

Use this in equation (4), $0 + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$

$$\frac{1}{2\lambda^2} = 1$$

$$\lambda^2 = \frac{1}{2}$$

$$\lambda = \frac{1}{\sqrt{2}} \quad \{\text{take } \lambda > 0\}$$

Then,

$$\begin{aligned} y^2 = z^2 \\ &= \frac{1}{2\lambda} \\ &= \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

So we have $y = z = \pm\frac{1}{(2)^{\frac{1}{4}}}$

Similarly, if $y = 0$, then x and $z \neq 0$.

Then, $x = z = \pm\frac{1}{(2)^{1/4}}$

And if z = 0, then x and $y \neq 0$.

Then by the symmetry of the equations (1), (2), and (3), $x = y = \pm\frac{1}{(2)^{1/4}}$

If $x, y, \text{ and } z \neq 0$, then $\lambda \neq 0$.

Then by the equations (1), (2), and (3) is $x^2 = y^2 = z^2 = \frac{1}{2\lambda}$.

Use this in (4), we have

$$x^4 + y^4 + z^4 = 1$$

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\frac{3}{4\lambda^2} = 1$$

$$\lambda^2 = \frac{3}{4}$$

$$\lambda = \frac{\sqrt{3}}{2}$$

Then by the equation $x^2 = y^2 = z^2 = \frac{1}{2\lambda}$, we have $x^2 = y^2 = z^2 = \frac{1}{\sqrt{3}}$

$$x = y = z = \pm \frac{1}{(3)^{\frac{1}{4}}}$$

Also, $\lambda \neq 0$ because if $\lambda = 0$, then by the equations (1), (2), and (3), $x = y = z = 0$.

But this is not possible because of the 4th equation $x^4 + y^4 + z^4 = 1$.

Also, if $x = 0$ an $y = 0$ then $z = \pm 1$

Similarly, if x an $z = 0$, then $y = \pm 1$.

And if y and $z = 0$, then $x = \pm 1$

Use the above information, the possible extreme points of f are as follows:

$$\left(0,\pm\frac{1}{(2)^{1/4}},\pm\frac{1}{(2)^{1/4}}\right),\left(\pm\frac{1}{(2)^{1/4}},0,\pm\frac{1}{(2)^{1/4}}\right),\left(\pm\frac{1}{(2)^{1/4}},\pm\frac{1}{(2)^{1/4}},0\right)$$

$$\left(\pm\frac{1}{(3)^{1/4}},\pm\frac{1}{(3)^{1/4}},\pm\frac{1}{(3)^{1/4}}\right),\ (0,0,\pm1)(0,\pm1,0),\ (\pm1,0,0)$$

Evaluate $f(x,y,z)=x^2+y^2+z^2$ at these points,

$\left(0,\pm\frac{1}{(2)^{1/4}},\pm\frac{1}{(2)^{1/4}}\right),\left(\pm\frac{1}{(2)^{1/4}},0,\pm\frac{1}{(2)^{1/4}}\right),\left(\pm\frac{1}{(2)^{1/4}},\pm\frac{1}{(2)^{1/4}},0\right)$ we have

$$\begin{aligned}
f\left(0,\pm\frac{1}{(2)^{1/4}},\pm\frac{1}{(2)^{1/4}}\right)&=f\left(\pm\frac{1}{(2)^{1/4}},0,\pm\frac{1}{(2)^{1/4}}\right)\\
&=f\left(\pm\frac{1}{(2)^{1/4}},\pm\frac{1}{(2)^{1/4}},0\right)\\
&=\left(\frac{1}{(2)^{1/4}}\right)^2+\left(\frac{1}{(2)^{1/4}}\right)^2\\
&=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\\
&=\sqrt{2}
\end{aligned}$$

The point $\left(\pm\frac{1}{(3)^{1/4}},\pm\frac{1}{(3)^{1/4}},\pm\frac{1}{(3)^{1/4}}\right)$, the value of $f(x,y,z)=x^2+y^2+z^2$ is as follows:

$$\begin{aligned}
f\left(\pm\frac{1}{(3)^{1/4}},\pm\frac{1}{(3)^{1/4}},\pm\frac{1}{(3)^{1/4}}\right)&=\left(\pm\frac{1}{(3)^{1/4}}\right)^2+\left(\pm\frac{1}{(3)^{1/4}}\right)^2+\left(\pm\frac{1}{(3)^{1/4}}\right)^2\\
&=\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\\
&=\sqrt{3}
\end{aligned}$$

And finally the points $(0,0,\pm1),(0,\pm1,0),(\pm1,0,0)$, the value of $f(x,y,z)=x^2+y^2+z^2$ are as follows:

$$f(0,0,\pm1) = f(0,\pm1,0)$$
$$= f(\pm1,0,0)$$
$$= (\pm1)^2 + (0)^2 + (0)^2$$
$$= 1$$

Of all the functional values $\sqrt{2}, \sqrt{3}, 1$, the maximum value is $\sqrt{3}$, and the minimum value is 1.

Therefore, the maximum value of f is $\boxed{\sqrt{3}}$ and the minimum value f is $\boxed{1}$.

Answer 12E.

$$f(x,y,z) = x^4 + y^4 + z^4$$
$$g(x,y,z) = x^2 + y^2 + z^2 = 1$$

By Lagrange's method of multipliers, we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $\quad g(x,y,z) = k$

i.e. $\quad f_x = \lambda g_x$

$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$
$$g(x,y,z) = k$$

i.e. $\quad 4x^3 = \lambda 2x$

$$4y^3 = \lambda 2y$$
$$4z^3 = \lambda 2z$$
$$x^2 + y^2 + z^2 = 1$$

i.e. $\quad 2x^3 = \lambda x$ -------------- (1)

$2y^3 = \lambda y$ -------------- (2)

$2z^3 = \lambda z$ -------------- (3)

$x^2 + y^2 + z^2 = 1$ -------------- (4)

Now $\lambda \neq 0$ because if $\lambda = 0$, $x = y = z = 0$, which is not possible because of (4)

If $x = 0$, $y, z \neq 0$ then

$$\lambda = 2y^2 = 2z^2$$

i.e. $\quad y^2 = z^2 = \dfrac{\lambda}{2}$

Using this in (4): -

$$0+\frac{\lambda}{2}+\frac{\lambda}{2}=1$$

i.e. $\lambda = 1$

Then $y^2 = z^2 = \frac{1}{2}$

And $y = z = \pm\frac{1}{\sqrt{2}}$

Similarly if $y = 0,\ x, z \neq 0$

Then $x = z = \pm\frac{1}{\sqrt{2}}$

If $z = 0, x, y \neq 0$

Then $x = y = \pm\frac{1}{\sqrt{2}}$

Also if $x = y = 0$ and $z \neq 0$ then $z = \pm 1$
If $y = z = 0$ and $x \neq 0$ then $x = \pm 1$
If $x = z = 0$ and $y \neq 0$ then $y = \pm 1$

Also if $x, y, z \neq 0$ then

$$x^2 = y^2 = z^2 = \frac{\lambda}{2} \qquad \{\text{From (1), (2), (3)}\}$$

Using this in (4)

$$\frac{\lambda}{2}+\frac{\lambda}{2}+\frac{\lambda}{2}=1$$

i.e. $\lambda = \frac{2}{3}$

i.e. $x = y = z = \pm\frac{1}{\sqrt{3}}$

Then all the possible extreme points of "f" are

$$\left(0,\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right),\ \left(\pm\frac{1}{\sqrt{2}},0,\pm\frac{1}{\sqrt{2}}\right),\ \left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},0\right)$$

$$(0,0,\pm 1),\ (0,\pm 1,0),\ (\pm 1,0,0),\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$$

Evaluating "f" on these points

$$f\left(0,\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=f\left(\pm\frac{1}{\sqrt{2}},0,\pm\frac{1}{\sqrt{2}}\right)=f\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},0\right)=\frac{1}{2}$$

$$f(0,0,\pm1)=f(0,\pm1,0)=f(\pm1,0,0)=1$$

$$f\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)=\frac{1}{3}$$

Hence the maximum value of "f" is $\boxed{1}$

And the minimum value is $\boxed{\frac{1}{3}}$

Answer 13E.

Consider the functions

$$f(x,y,z,t)=x+y+z+t$$

$$g(x,y,z,t)=x^2+y^2+z^2+t^2=1$$

By Lagrange's method of multipliers we find all *x, y, z, t* and ? such that

$$\vec{\nabla}f(x,y,z,t)=\lambda\vec{\nabla}g(x,y,z,t)$$

and $g(x,y,z,t)=k$

i.e., $f_x=\lambda g_x$

$$1=2\lambda x \quad \cdots\cdots(1)$$

$$f_y=\lambda g_y$$

$$1=2\lambda y \quad \cdots\cdots(2)$$

$$f_z=\lambda g_z$$

$$1=2\lambda z \quad \cdots\cdots(3)$$

$$f_t=\lambda g_t$$

$$1=2\lambda t \quad \cdots\cdots(4)$$

$$g(x,y,z,t)=k$$

$$x^2+y^2+z^2+t^2=1 \quad \cdots\cdots(5)$$

Now (1), (2), (3), (4) gives

$$x=\frac{1}{2\lambda},\ \ y=\frac{1}{2\lambda},\ \ z=\frac{1}{2\lambda},\ t=\frac{1}{2\lambda}$$

(As (x,y,z), because of (1), (2), (3), (4))

Using this in (5)

$$\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1$$

i.e., $\lambda^2=1 \ \Rightarrow \ \lambda=\pm 1$

i.e., $x=y=z=t=\pm\frac{1}{2}$

Then all the possible extreme points are

$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{2}\right)$

$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)$, $\left(\frac{-1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{-1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{2}\right)$, $\left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)$

$\left(\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}\right)$, $\left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}\right)$

$\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{2}\right)$, $\left(\frac{-1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}\right)$, $\left(\frac{-1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}\right)$

$\left(\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right)$, $\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right)$

Evaluating "f" on these extreme points

$$f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=2$$

$$f\left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=f\left(\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{2}\right)=f\left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{2}\right)=f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)=1$$

$$f\left(\frac{-1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{2}\right)=f\left(\frac{-1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{2}\right)=f\left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)=0$$

$$f\left(\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{2}\right)=f\left(\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}\right)=f\left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}\right)=0$$

$$f\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{2}\right)=f\left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)=f\left(\frac{-1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}\right)=f\left(\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right)=-1$$

and $f\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right)=-2$

Therefore the maximum value of "f" is $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=\boxed{2}$

And the minimum value of "f" is $f\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right)=\boxed{-2}$

Answer 14E.

$$f(x_1,x_2,x_3,\ldots\ldots\ldots,x_n)=x_1+x_2+\ldots\ldots+x_n$$
$$g(x_1,x_2,x_3,\ldots\ldots\ldots,x_n)=x_1^2+x_2^2+\ldots\ldots x_3^2=1$$

By Lagrange's method of multipliers we find all $x_1,x_2,\ldots\ldots,x_n$ and λ such that

$$\vec{\nabla}f(x_1,x_2,\ldots\ldots,x_n)=\lambda\vec{\nabla}g(x_1,x_2,\ldots\ldots,x_n)$$

And $g(x_1,x_2,\ldots\ldots,x_n)=k$

i.e.

$$f_{x_1}=\lambda g_{x_1}$$
$$f_{x_2}=\lambda g_{x_2}$$
$$\vdots$$
$$f_{x_n}=\lambda g_{x_n}$$
$$g(x_1,x_2,\ldots\ldots,x_n)=k$$

i.e. $1 = 2\lambda x_1$ ------------------- (1)

$1 = 2\lambda x_2$ ------------------- (2)

$\vdots$ $\vdots$

$1 = 2\lambda x_n$ ------------------- (n)

$x_1^2 + x_2^2 + \dots\dots\dots + x_n^2 = 1$ ------------------- (n+1)

Now $\lambda \neq 0$ and $x_1, x_2, \dots\dots\dots, x_n \neq 0$

Then we have $x_1 = x_2 = x_3 \dots\dots\dots = x_n = \frac{1}{2\lambda}$

Using this in equation (n + 1)

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \dots\dots + \frac{1}{4\lambda^2} = 1$$

i.e. $\frac{n}{4} = \lambda^2$

i.e. $\lambda = \pm\frac{\sqrt{n}}{2}$

Then $x_1 = x_2 = \dots\dots\dots = x_n = \pm\frac{1}{\sqrt{n}}$

Then all the possible extreme points are

$$\left(\pm\frac{1}{\sqrt{n}}, \pm\frac{1}{\sqrt{n}} \dots\dots\dots\dots, \pm\frac{1}{\sqrt{n}}\right)$$

On evaluating "f" on these extreme values we find the maximum value of "f" is

$$f\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots\dots, \frac{1}{\sqrt{n}}\right) = \frac{n}{\sqrt{n}}$$

$$= \boxed{\sqrt{n}}$$

And the minimum value of "f" is

$$f\left(-\frac{1}{\sqrt{n}}, \frac{-1}{\sqrt{n}}, \dots\dots\dots\dots, \frac{-1}{\sqrt{n}}\right) = \frac{-n}{\sqrt{n}}$$

$$= \boxed{-\sqrt{n}}$$

Answer 15E.

$$f(x,y,z) = x + 2y$$

$$g(x,y,z) = x + y + z = 1$$

$$h(x,y,z) = y^2 + z^2 = 1$$

By Lagrange's method of multipliers we find all x, y, z λ and μ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) + \mu \vec{\nabla} h(x,y,z)$$

$$g(x,y,z) = k$$

$$h(x,y,z) = c$$

i.e. $1 = \lambda + \mu(0)$ ------------ (1)

$2 = \lambda + \mu 2y$ ------------ (2)

$0 = \lambda + \mu 2z$ ------------ (3)

$x + y + z = 1$ ------------ (4)

$y^2 + z^2 = 4$ ------------ (5)

Equation (1) gives $\lambda = 1$

Then from (2); $1 = 2\mu y$

From (3): $-1 = 2\mu z$

i.e. $y = \dfrac{1}{2\mu}$ $z = \dfrac{-1}{2\mu}$

Using this in equation (5)

$$\frac{1}{4\mu^2} + \frac{1}{4\mu^2} = 4$$

i.e. $\mu^2 = \dfrac{1}{8}$

$$\Rightarrow \mu = \pm \frac{1}{2\sqrt{2}}$$

Then $y = \pm\sqrt{2}$, $z = \mp\sqrt{2}$

Using these values in equation (4);

When $y = \sqrt{2}$, $z = -\sqrt{2}$, $x = 1$

When $y = -\sqrt{2}$, $z = \sqrt{2}$, $x = 1$

Therefore the possible extreme points of "f" are

$$\left(1, \sqrt{2}, -\sqrt{2}\right), \left(1, -\sqrt{2}, \sqrt{2}\right)$$

Evaluating "f" on these points

$$f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$$

$$f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$$

Hence maximum value of "f" is $f\left(1, \sqrt{2}, -\sqrt{2}\right) = \boxed{1 + 2\sqrt{2}}$

And minimum value of "f" is $f\left(1, -\sqrt{2}, \sqrt{2}\right) = \boxed{1 - 2\sqrt{2}}$

Answer 16E.

$$f(x,y,z) = 3x - y - 3z$$
$$g(x,y,z) = x + y - z = 0$$
$$h(x,y,z) = x^2 + 2z^2 = 1$$

By Lagrange's method of multipliers we find all x, y, z, λ and μ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) + \mu \vec{\nabla} h(x,y,z)$$
$$g(x,y,z) = k$$
$$h(x,y,z) = c$$

i.e. $3 = \lambda + 2x\mu$ ----------------- (1)

$-1 = \lambda$ ----------------- (2)

$-3 = -\lambda + \mu 4z$ ----------------- (3)

$x + y - z = 0$ ----------------- (4)

$x^2 + 2z^2 = 1$ ----------------- (5)

From equation (2) $\lambda = -1$

Then from equations (1) and (3)

$$2\mu x = 4$$
$$4\mu z = -4$$

i.e. $x = \dfrac{2}{\mu}$ and $z = \dfrac{-1}{\mu}$ $\{\mu \neq 0\}$

Using these values in equation (5)

$$\frac{4}{\mu^2} + \frac{2}{\mu^2} = 1$$

i.e. $\mu^2 = 6$

i.e. $\mu = \pm\sqrt{6}$

Then $x = \pm\dfrac{2}{\sqrt{6}}$, $z = \mp\dfrac{1}{\sqrt{6}}$

Using these values in equation (4)

$$y = z - x$$

$y = -\dfrac{3}{\sqrt{6}}$; when $z = \dfrac{-1}{\sqrt{6}}$, $x = \dfrac{2}{\sqrt{6}}$

And $y = \dfrac{3}{\sqrt{6}}$; when $z = \dfrac{1}{\sqrt{6}}$, $x = \dfrac{-2}{\sqrt{6}}$

Then possible extreme points of "f" are

$$\left(\frac{2}{\sqrt{6}},\frac{-3}{\sqrt{6}},\frac{-1}{\sqrt{6}}\right),\left(\frac{-2}{\sqrt{6}},\frac{3}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

Evaluating "f" on these extreme points

$$f\left(\frac{2}{\sqrt{6}},\frac{-3}{\sqrt{6}},\frac{-1}{\sqrt{6}}\right)=\frac{6}{\sqrt{6}}+\frac{3}{\sqrt{6}}+\frac{3}{\sqrt{6}}$$
$$=\frac{12}{\sqrt{6}}$$
$$=2\sqrt{6}$$

And $$f\left(\frac{-2}{\sqrt{6}},\frac{3}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)=\frac{-6}{\sqrt{6}}-\frac{3}{\sqrt{6}}-\frac{3}{\sqrt{6}}$$
$$=\frac{-12}{\sqrt{6}}$$
$$=-2\sqrt{6}$$

Hence maximum value of "f" is $\boxed{2\sqrt{6}}$

And minimum value of "f" is $\boxed{-2\sqrt{6}}$

Answer 17E.

$$f(x,y,z)=yz+xy$$
$$g(x,y,z)=xy=1$$
$$h(x,y,z)=y^2+z^2=1$$

By Lagrange's method of multipliers we find all x, y, z, λ and μ such that

$$\vec{\nabla}f(x,y,z)=\lambda\vec{\nabla}g(x,y,z)+\mu\vec{\nabla}h(x,y,z)$$
$$g(x,y,z)=k$$
$$h(x,y,z)=c$$

i.e. $y=\lambda y$ ----------------- (1)

$z+x=\lambda x+2\mu y$ ----------------- (2)

$y=\mu 2z$ ----------------- (3)

$xy=1$ ----------------- (4)

$y^2+z^2=1$ ----------------- (5)

Now $x, y \neq 0$ (because of (3))

Then from equation (1), $\lambda = 1$

Then from equation (2), $z + x = x + 2\mu y$

i.e. $z = 2\mu y$

$$\Rightarrow \frac{z}{y} = 2\mu \text{ ---------------- (6)}$$

From equation (3) $y = 2\mu z$

$$\Rightarrow \frac{z}{y} = \frac{1}{2\mu} \text{ -------------- (7)}$$

From (6) and (7); $2\mu = \frac{1}{2\mu}$

i.e. $\mu^2 = \frac{1}{4}$

i.e. $\mu = \pm\frac{1}{2}$

Then $z = \pm y$

Then from equation (5); $y^2 + y^2 = 1$

i.e. $2y^2 = 1$

i.e. $y = \pm\frac{1}{\sqrt{2}}$

i.e. $\pm z = y = \pm\frac{1}{\sqrt{2}}$

Also from equation (4); $x = \frac{1}{y} = \pm\sqrt{2}$

Then all the possible extreme points are

$$\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

Evaluating "f" on these points

$$f\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{3}{2}$$

$$f\left(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2}$$

Hence the maximum value of "f" is $\boxed{\frac{3}{2}}$

And the minimum value of "f" is $\boxed{\frac{1}{2}}$

Answer 18E.

Recall the method of Lagrange multipliers,

To find the maximum or minimum of a function subject to the constraints $g(x,y,z)=k$,

$h(x,y,z)=c$ $\nabla g \neq 0, \nabla h \neq 0$

(a) find all the values of *x*, *y*, *z* , λ and μ such that

$$\nabla f(x,y,z)=\lambda \nabla g(x,y,z)+\mu \nabla h(x,y,z)$$

$$\text{and } g(x,y,z)=k,\ h(x,y,z)=c$$

(b) Evaluate the value of *f* at these points. The largest of these values is maximum and the smallest of these values is minimum.

Consider the function

$f(x, y, z) = x^2 + y^2 + z^2$, ... (1)

And the constraints are

$g(x, y, z) = x - y - 1$ ...... (a)

$h(x, y, z) = y^2 - z^2 - 1$ ...... (b)

The vector equation $\nabla f(x,y,z)=\lambda \nabla g(x,y,z)+\mu \nabla h(x,y,z)$ in terms of its components is f are

$f_x = \lambda g_x + \mu h_x$ $f_y = \lambda g_y + \mu h_y$ $f_z = \lambda g_z + \mu h_z$ $g(x,y,z)=1$,

$h(x,y,z)=1$

Differentiate (1) with respect to *x* partially.

$$2x = \lambda + 0$$

$2x = \lambda$ ...... (2)

Differentiate (1) with respect to *y* partially.

$$2y = \lambda(-1) + \mu(2y)$$

$2y = -\lambda + 2\mu y$ ...... (3)

Differentiate (1) with respect to *z* partially.

$$2z = 0 + \mu(-2z)$$

$2z = -2\mu z$ ...... (4)

Solve all these equations for x, y, z, λ and μ

Since from (*a*)

$$x - y = 1$$
$$x = y + 1$$
$$\frac{\lambda}{2} = y + 1 \quad \text{From (2)} \quad 2x = \lambda \Rightarrow x = \frac{\lambda}{2}$$
$$y = \frac{\lambda}{2} - 1$$

Plug in $y = \frac{\lambda}{2} - 1$ in equation (3)

$$2y = -\lambda + 2\mu y$$
$$2\left(\frac{\lambda}{2} - 1\right) = -\lambda + 2\mu\left(\frac{\lambda}{2} - 1\right)$$
$$\lambda - 2 = -\lambda + \mu\lambda - 2$$
$$2\lambda - 2 = \mu\lambda - 2$$
$$2\lambda = \mu\lambda$$
$$\mu = 2$$

Therefore $\boxed{\mu = 2}$

To obtain the *z* value plug in $\mu = 2$ in equation (4)

Then

$$2z = -2\mu z$$
$$2z = -2(2)z$$
$$2z = -4z$$
$$\Rightarrow z = 0$$

Thus $\boxed{z = 0}$

To obtain the *y* value, plug in $\boxed{z=0}$ in equation (*b*)

Then

$$\begin{aligned} y^2 - z^2 &= 1 \\ y^2 - 0 &= 1 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

Therefore the *y* values are $\boxed{-1 \text{ and } 1}$

Next, Substitute $y = \pm 1$ in equation (*a*)

Then

$$\begin{aligned} x - y &= 1 \\ x - 1 &= 1 \\ x &= 1 + 1 \\ x &= 2 \end{aligned} \quad \text{And} \quad \begin{aligned} x - y &= 1 \\ x - (-1) &= 1 \\ x + 1 &= 1 \\ x &= 0 \end{aligned}$$

Therefore the corresponding *x* values are

$$\boxed{\begin{aligned} &\text{if } y = 1 \text{ then } x = 2, \\ &\text{if } y = -1 \text{ then } x = 0 \end{aligned}}$$

Finally to solve for λ, use *x* values in equation $2x = \lambda$

Then

$$\begin{aligned} 2x &= \lambda \\ 2(2) &= \lambda \\ \lambda &= 4 \end{aligned} \quad \text{And} \quad \begin{aligned} 2x &= \lambda \\ 2(0) &= \lambda \\ \lambda &= 0 \end{aligned}$$

Therefore the solutions are $(x, y, z) = \boxed{(2,1,0) \text{ and } (0,-1,0)}$

Now find the maximum and minimum values of f at these two points.

$$\begin{aligned} f_{(2,1,0)} &= x^2 + y^2 + z^2 \\ &= 2^2 + 1^2 + 0^2 \\ &= 4 + 1 + 0 \\ &= \boxed{5} \end{aligned}$$

And

$$\begin{aligned} f_{(0,-1,0)} &= x^2 + y^2 + z^2 \\ &= 0^2 + (-1)^2 + 0^2 \\ &= 0 + 1 + 0 \\ &= \boxed{1} \end{aligned}$$

Therefore

The function f has maximum at the point $(2,1,0)$ and the maximum value is $\boxed{5}$.

The function f has minimum at the point $(0,-1,0)$ and the minimum value is $\boxed{1}$.

Answer 19E.

Given $f(x,y) = x^2 + y^2 + 4x - 4y, x^2 + y^2 \leq 9$

Finding the partial derivatives f_x and f_y we get.

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(x^2 + y^2 + 4x - 4y\right) \\ &= 2x + 4 \end{aligned}$$

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\left(x^2 + y^2 + 4x - 4y\right) \\ &= 2y - 4 \end{aligned}$$

Equate f_x to 0 and f_y to 0 to find the critical points.

$$\begin{aligned} 2x + 4 &= 0 \\ x &= -2 \end{aligned}$$

$$\begin{aligned} 2y - 4 &= 0 \\ y &= 2 \end{aligned}$$

Thus, the critical point is obtained as $(-2,2)$.

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0, z_0) on the smooth constraint curve $g(x,y,z)=k$.
If $\nabla g(x, y, z) \neq 0$, then there is a real number λ such that
$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

Let $g(x, y) = x^2 + y^2 = 9$.

Find $\nabla g(x, y)$.

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle$$
$$= \langle 2x, 2y \rangle$$

Then, $\nabla f = \lambda \nabla g$ or $\langle 2x+4, 2y-4 \rangle = \lambda \langle 2x, 2y \rangle$.

On equating the like terms, we get $2x + 4 = 2\lambda x$ or $x = \dfrac{-2}{1-\lambda}$ and

$2y - 4 = 2\lambda y$ or $y = \dfrac{2}{1-\lambda}$.

Replace x with $\dfrac{-2}{1-\lambda}$ and y with $\dfrac{2}{1-\lambda}$ in $x^2 + y^2 = 9$ and solve for λ.

$$\left(\frac{-2}{1-\lambda}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 = 9$$
$$\frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 9$$
$$8 = 9(1-\lambda)^2$$
$$(1-\lambda)^2 = \frac{8}{9}$$
$$\lambda = 1 \pm \frac{2\sqrt{2}}{3}$$

On substituting $1 + \dfrac{2\sqrt{2}}{3}$ for λ and $1 - \dfrac{2\sqrt{2}}{3}$ for λ in the equations for x and y, we get

$\left(\dfrac{3}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right)$ and $\left(-\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$.

Now, find $f(-2,2)$, $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ and $f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$.

$$f(-2,2) = (-2)^2 + 2^2 + 4(-2) - 4(2)$$
$$= -8$$
$$f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \left(\frac{3}{\sqrt{2}}\right)^2 + \left(-\frac{3}{\sqrt{2}}\right)^2 + 4\left(\frac{3}{\sqrt{2}}\right) - 4\left(-\frac{3}{\sqrt{2}}\right)$$
$$= 9 + 12\sqrt{2}$$
$$f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \left(-\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{3}{\sqrt{2}}\right)^2 + 4\left(-\frac{3}{\sqrt{2}}\right) - 4\left(\frac{3}{\sqrt{2}}\right)$$
$$= 9 - 12\sqrt{2}$$

Thus, we note that the maximum value of f is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2}$ and the minimum value is $f(-2, 2) = -8$.

Answer 20E.

Now $f(x,y) = 2x^2 + 3y^2 - 4x - 5$

Then $f_x = 4x - 4$

$f_y = 6y$

We find critical points by setting $f_x = 0$, $f_y = 0$

i.e. $4(x-1) = 0$

And $6y = 0$

i.e. $x = 1$

And $y = 0$

The critical points is $(1,0)$

Now we use Lagrange's method of multipliers to find extreme values of "f" subject to the constraint $x^2 + y^2 = 16$

We solve the equations

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$$

And $g(x,y) = 16$

i.e. $4x - 4 = \lambda 2x$ ------------ (1)

$6y = \lambda 2y$ ------------ (2)

$x^2 + y^2 = 16$ ------------ (3)

If y = 0, then from equation (3) $x = \pm 4$
If $y \neq 0$, then from equation (2) $\lambda = 3$
Using this in equation (1) we get x = - 2
And then from (3) $y = \pm 2\sqrt{3}$

Then all the extreme points of "f" are

$$(4,0),\ (-4,0),\ \left(-2, 2\sqrt{3}\right),\ \left(-2, -2\sqrt{3}\right)$$

We comparing the value of "f" at the critical point with the extreme values on the boundary

$$f(1,0) = -7$$
$$f(4,0) = 11$$
$$f(-4,0) = 43$$
$$f\left(-2, 2\sqrt{3}\right) = 47$$
$$f\left(-2, -2\sqrt{3}\right) = 47$$

Therefore the maximum value of "f" is $\boxed{47}$
And the minimum value is $\boxed{-7}$

Answer 21E.

$$f(x,y) = e^{-xy}$$

First we find the critical point

$$f_x = -ye^{-xy}$$
$$f_y = -xe^{-xy}$$

We find the critical point by setting $f_x = 0$, $f_y = 0$

i.e. $ye^{-xy} = 0$

And $xe^{-xy} = 0$

Since $e^{-xy} \neq 0$, then the critical point is (0, 0)

Now we use Lagrange's method of multipliers to find the extreme values of "f" subject to the constraint $x^2 + 4y^2 = 1$
We solve the equations

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$$
$$g(x,y) = 1$$

i.e. $-ye^{-xy} = \lambda 2x$ ----------- (1)

$-xe^{-xy} = \lambda 8y$ ----------- (2)

$x^2 + 4y^2 = 1$ ------------ (3)

Then $\lambda \neq 0$ because if $\lambda = 0$, $x = 0$, $y = 0$ which is not possible because of equation (3)

Now from equation (1) $\quad -xye^{-xy} = \lambda 2x^2$

And from equation (2) $\quad -xye^{-xy} = \lambda 8y^2$

i.e. $\quad \lambda 2x^2 = \lambda 8y^2$

Since $\lambda \neq 0$ then $x^2 = 4y^2$

Using this in equation (3); $\quad 4y^2 + 4y^2 = 1$

$8y^2 = 1$

i.e. $\quad y = \pm\frac{1}{2\sqrt{2}}$

Then $\quad x = \pm\frac{1}{\sqrt{2}}$

Then all the extreme points are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$$

We compare the value of "f" at the critical point with that obtained at the extreme points

$$f(0,0) = 1$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right) = e^{-1/4}$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right) = e^{1/4}$$

Hence maximum value of "f" is $\boxed{e^{1/4}}$ and the minimum value is $\boxed{e^{-1/4}}$

Answer 24E.

a)

Consider the function

$$f(x,y)=x^3+y^3+3xy$$

Subject to constraint

$$g(x,y)=(x-3)^2+(y-3)^2=9$$

First load the package with(plots);

> *with*(*plots*);

Enter the function f in maple as follows.

f(x, y):=x^3+y^3+3*x*y;

Maple input and output:

> $f(x, y) := x^3 + y^3 + 3 \cdot x \cdot y$

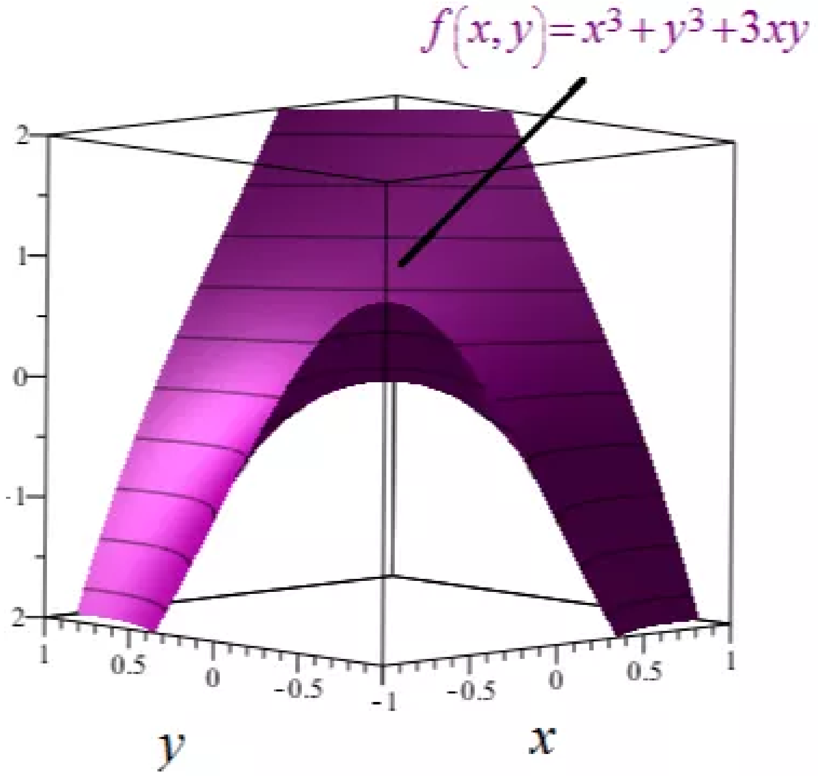
$f := (x, y) \rightarrow x^3 + y^3 + 3\,x\,y$

To plot f, use the following command.

plot3d(f(x, y),x=-1..1,y=-1..1, style = patchcontour, color = purple, axes = boxed);

The maple output is

> *plot3d*(*f*(*x*,*y*), *x*=−1 ..1, *y*=−1 ..1, *style*=*patchcontour*, *color*=*purple*, *axes*=*boxed*);



Enter the function g as follows.

g(x, y):=(x-3)^2+(y-3)^2-9;

The maple output is

> $g(x, y) := (x-3)^2 + (y-3)^2 - 9;$

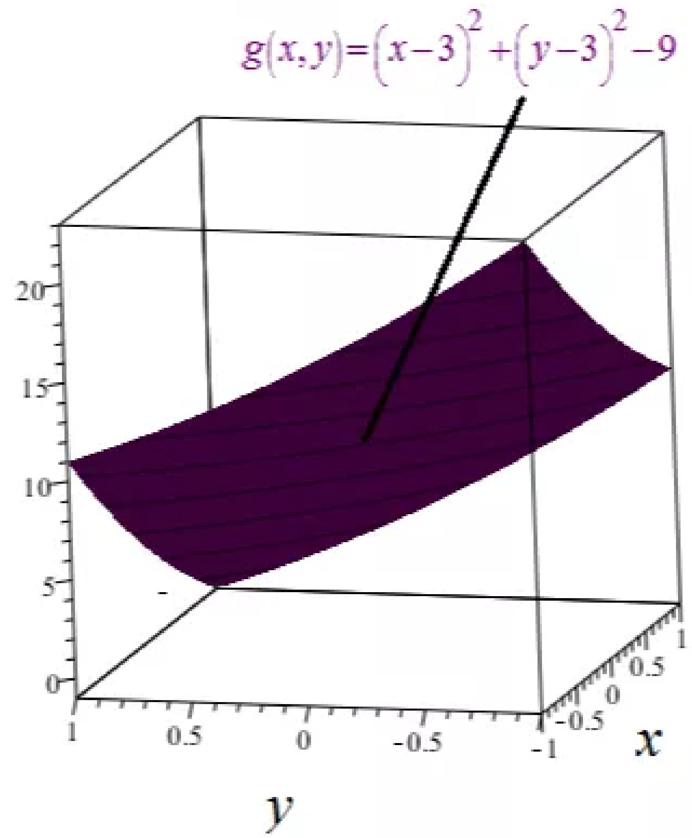
$g := (x, y) \rightarrow (x-3)^2 + (y-3)^2 - 9$

To plot g, use the following command.

Plot3d(g(x, y),x=-1..1,y=-1..1, style = patch contour, color = blue, axes = boxed);

The maple input and output as follows:

> *plot3d*(*g*(*x*,*y*), *x* = −1 ..1, *y* = −1 ..1, *style* = *patchcontour*, *color* = *purple*, *axes* = *boxed*);

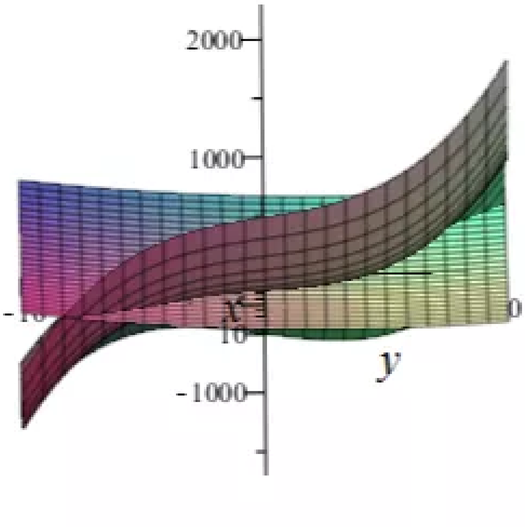


Enter the following maple command to get the graph of both *f* and *g* on the same window.

plot3d([f(x,y),g(x,y)],x=-10..10,y=-10..10,axes=normal);

Maple input and output:

> *plot3d*([*f*(*x*, *y*), *g*(*x*, *y*)], *x* = −10 ..10, *y* = −10 ..10, *axes* = *normal*);



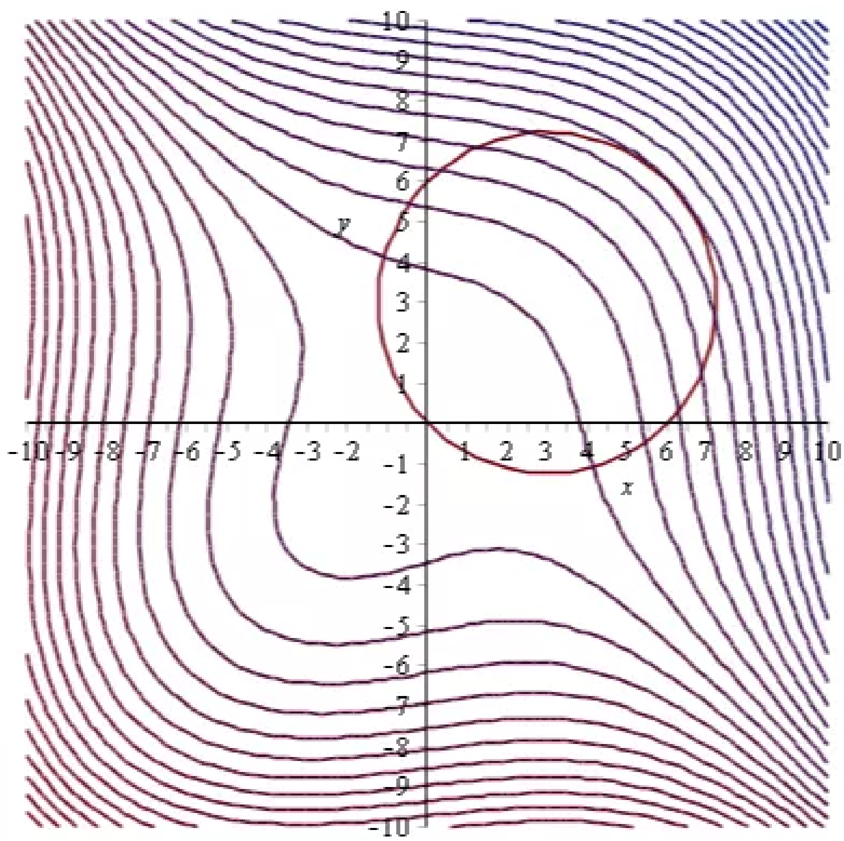
To find the maximum and minimum values of the function *f* subject to the constraint *g* , use contours.

To display the contours, use the following command.

display(implicitplot(g(x,y)=9,x=-10..10,y=-10..10);contourplot(f(x,y),x=-10..10,y=-10..10,contours

Maple input and output:

> *display*(*implicitplot*(*g*(*x*, *y*) = 9, *x* = −10 ..10, *y* = −10 ..10), *contourplot*(*f*(*x*, *y*), *x* = −10 ..10, *y* = −10 ..10, *contours* = 40));



From the contour plot it is clear that, the function has a minimum value at nearly $(0.87, 0.87)$ and maximum value at $(5.12, 5.12)$.

Find the value of *f* at these points.

$f(0.87, 0.87) = (0.87)^3 + (0.87)^3 + 3(0.87)(0.87)$ Use $f(x, y) = x^3 + y^3 + 3xy$

$= 3.587706$ Use maple

$f(5.12, 5.12) = (5.12)^3 + (5.12)^3 + 3(5.12)(5.12)$ Use $f(x, y) = x^3 + y^3 + 3xy$

$= 347.078656$ Use maple

Therefore, the function has maximum value of f is $\boxed{347}$ and the minimum value of f is $\boxed{3.58}$.

(b)

Recall the method of Lagrange multipliers, to find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$);

1. Find all values of x, y, z and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$.

2. Evaluate f at all the points (x, y, z) that result from step (1).The largest of these values is the maximum value of f, the smallest is the minimum value of f

Use this to find the critical points and maximum and minimum values of f at these points.

Consider the function

$f(x, y) = x^3 + y^3 + 3xy$

Subject to constraint

$g(x, y) = (x-3)^2 + (y-3)^2 - 9$.

Using Lagrange multipliers,

$f_x = \lambda g_x$ ...... (1)

$f_y = \lambda g_y$ ...... (2)

$(x-3)^2 + (y-3)^2 - 9 = 0$ ...... (3)

Use CAS to solve the problem.

First load the package with(VectorCalculus);

Enter the function f in maple as follows.

f(x, y):=x^3+y^3+3*x*y;

Maple input and output:

> $f(x,y) := x^3 + y^3 + 3 \cdot x \cdot y;$

$f := (x,y) \rightarrow x^3 + y^3 + 3\,x\,y$

Enter the function g as follows.

g(x, y):=(x-3)^2+(y-3)^2;

The maple output is

> $g(x,y) := (x-3)^2 + (y-3)^2 - 9;$

$g := (x,y) \rightarrow (x-3)^2 + (y-3)^2 - 9$

Find ∇f and ∇g using maple.

Enter the command to find gradient of *f* as

df:=Del(f(x,y),[x,y]);

Maple output and input:

> $df := Del(f(x,y), [x,y]);$

$df := (3\,x^2 + 3\,y)\bar{e}_x + (3\,y^2 + 3\,x)\bar{e}_y$

Enter the command to find gradient of *f* as

dg:=Del(g(x,y),[x,y]);

Maple output and input:

> $dg := Del(g(x,y),[x,y]);$

$dg := (2x-6)\bar{e}_x + (2y-6)\bar{e}_y$

Uses solve command to solve the Lagrange equations.

Maple input:

solve({df[1]=lambda*dg[1],df[2]=lambda*dg[2],g(x,y)=9},{x,y,lambda}):

Sols:=map(allvalues,[%]);

> $solve(\{df[1] = \text{lambda}\cdot dg[1], df[2] = \text{lambda}\cdot dg[2], g(x,y) = 9\}, \{x, y, \text{lambda}\}) :$

> $sols := map(allvalues, [\%]);$

$$sols := \left[\left\{x = 3 - \frac{3}{2}\sqrt{2}, y = 3 - \frac{3}{2}\sqrt{2}, \lambda = \frac{21}{2} - \frac{33}{4}\sqrt{2}\right\}, \left\{x = 3 + \frac{3}{2}\sqrt{2}, y = 3 + \frac{3}{2}\sqrt{2}, \lambda = \frac{21}{2} + \frac{33}{4}\sqrt{2}\right\}\right]$$

Evaluate *f* these values.

Maple input:

f(3-(3/2)*sqrt(2), 3-(3/2)*sqrt(2));

Maple output:

> $f(3-(3/2)*\text{sqrt}(2), 3-(3/2)*\text{sqrt}(2));$

$$2\left(3-\frac{3}{2}\sqrt{2}\right)^3 + 3\left(3-\frac{3}{2}\sqrt{2}\right)^2$$

$$\xrightarrow{\text{at 5 digits}}$$

3.6732

Maple input:

f(3+(3/2)*sqrt(2), 3+(3/2)*sqrt(2));

Maple output:

> $f(3+(3/2)*\text{sqrt}(2), 3+(3/2)*\text{sqrt}(2));$

$$2\left(3+\frac{3}{2}\sqrt{2}\right)^3 + 3\left(3+\frac{3}{2}\sqrt{2}\right)^2$$

$$\xrightarrow{\text{at 5 digits}}$$

347.32

Therefore, the maximum value of $f(x,y)$ is $\boxed{347.32}$ and the minimum values of $f(x,y)$ is $\boxed{3.67}$.

Hence, the results in part (b) is approximately same as the result in part (a).

Answer 25E.

$$P = bL^{\alpha}\ K^{1-\alpha}$$
$$g(L,K) = mL + nK = p$$

We use Lagrange's method of multipliers to find all L, K and λ such that

$$\vec{\nabla}P = \lambda\vec{\nabla}g$$
$$g(L,K) = p$$

i.e. $b\,\alpha L^{\alpha-1}\,K^{1-\alpha} = \lambda m$ ----------- (1)

$b(1-\alpha)L^{\alpha}\,K^{-\alpha} = \lambda n$ ------------ (2)

$mL + nK = p$ ------------ (3)

When $\lambda \neq 0$ from equation (1) $b\alpha L^{\alpha}\,K^{1-\alpha} = \lambda mL$

From equation (2) $b(1-\alpha)L^{\alpha}K^{1-\alpha} = \lambda nK$

On adding $L^{\alpha}K^{1-\alpha}\left[b\alpha + b - b\alpha\right] = \lambda(mL + nK)$

$$b\,L^{\alpha}\,K^{1-\alpha} = \lambda(mL + nK)$$

Using equation (3); $b\,L^{\alpha}\,K^{1-\alpha} = \lambda p$

From equation (2); $\lambda = \dfrac{b(1-\alpha)L^{\alpha}\,K^{-\alpha}}{n}$

Then $b\,L^{\alpha}\,K^{1-\alpha} = \dfrac{b(1-\alpha)L^{\alpha}\,K^{-\alpha}p}{n}$

i.e. $K = \dfrac{(1-\alpha)p}{n}$

Then $L = \dfrac{p - nK}{m}$

$$= \frac{p - p + \alpha p}{m}$$

$$= \frac{\alpha p}{m}$$

If $\lambda = 0$ take $L = 0$

Then from equation (3) $K = \dfrac{p}{n}$

Take $K = 0$ then $L = \dfrac{p}{m}$

Then all the extreme points are $\left(\frac{\alpha p}{m}, \frac{(1-\alpha)p}{n}\right)$

$\left(0, \frac{p}{n}\right), \left(\frac{p}{m}, 0\right)$

Evaluating P on there extreme points

$$P\left(\frac{\alpha p}{m}, \frac{(1-\alpha)p}{n}\right) = b\left(\frac{\alpha p}{m}\right)^{\alpha}\left(\frac{(1-\alpha)p}{n}\right)^{1-\alpha}$$

$$P\left(0, \frac{p}{n}\right) = P\left(\frac{p}{m}, 0\right) = 0$$

Therefore maximum value of P is achieved when

$$L = \frac{\alpha p}{m}, \text{ and } \quad K = \frac{(1-\alpha)p}{n}$$

Hence prove

Answer 26E.

$$C(L,k) = mL + nk$$
$$g(L,k) = b\,L^{\alpha}\,k^{1-\alpha} = Q$$

By Lagrange's method we find L, k and λ such that

$$\vec{\nabla}C(L,k) = \lambda\,\vec{\nabla}g(L,k)$$
$$g(L,k) = Q$$

i.e. $m = \lambda\,\alpha b\,L^{\alpha-1}k^{1-\alpha}$ ----------- (1)

$n = \lambda b(1-\alpha)L^{\alpha}k^{-\alpha}$ ----------- (2)

$b\,L^{\alpha}\,k^{1\alpha} = Q$ ----------- (3)

From equation (1); $mL = \lambda\alpha b\,L^{\alpha}\,k^{1-\alpha}$ --------------- (4)

And from equation (2); $nk = \lambda b(1-\alpha)L^{\alpha}\,k^{1-\alpha}$ --------------- (5)

Equation (4) and (5) gives

$$L = \frac{n\alpha}{m(1-\alpha)}k$$

Using this value of L in equation (3)

$$b\left[\frac{n\alpha}{m(1-\alpha)}\right]^{\alpha} k^{1-\alpha} = Q$$

i.e. $k^{1-\alpha} = \left[\frac{m(1-\alpha)}{n\alpha}\right]^{\alpha} \cdot \frac{Q}{b}$

Or $k = \left[\frac{m(1-\alpha)}{n\alpha}\right]^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}$

Then $L = \frac{n\alpha}{m(1-\alpha)} \cdot \left[\frac{m(1-\alpha)}{n\alpha}\right]^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}$

$$= \left[\frac{m(1-\alpha)}{n\alpha}\right]^{\frac{2\alpha-1}{1-\alpha}} \cdot \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}$$

Also from equation (1) and (2)

$$\frac{m}{\alpha} = \frac{n}{1-\alpha}$$

i.e. $m(1-\alpha) = n\alpha$

This gives $L = k = \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}$

Now the nature of the problem dictates that there must be an absolute minimum cost at the extreme points of C(L, m), so it must occur at $L = k = \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}$

Hence when $\boxed{L = \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}}$

And $\boxed{k = \left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}}$

The cost function will be minimum. And the minimum cost will be

$$C(L,k)_{\min} = (m+n)\left(\frac{Q}{b}\right)^{\frac{1}{1-\alpha}}$$

Answer 27E.

Let $A(x,y) = xy$

$g(x,y) = 2x + 2y = p$

Using Lagrange's method we find x, y and λ such that

$$\vec{\nabla} A(x,y) = \lambda \vec{\nabla} g(x,y)$$

$$g(x,y) = p$$

i.e. $y = 2\lambda$ ----------- (1)

$x = 2\lambda$ ------------ (2)

$2x + 2y = p$ ------------ (3)

From equation (1) and (2) $x = y$

Using this in (3); $x = y = \frac{p}{4}$

Then $\left(\frac{p}{4},\frac{p}{4}\right)$ is the only extreme point of A(x, y)

Now the physical nature of the problem says that there must be an absolute maximum area which has to occur at the extreme point of A(x, y) so it must occur at $x = y = p/4$

And hence the rectangle with given perimeter will have maximum area if it is a square.

Answer 28E.

Assume x, y, z are the lengths of the sides of a triangle.

The perimeter of the triangle is calculated as follows:

$$2s = x + y + z$$
$$s = \frac{x+y+z}{2}$$

Use herons formula, the area of the triangle with the side's lengths $x, y,$ and z is defined as follows:

$$A = \sqrt{s(s-x)(s-y)(s-z)}.$$

To maximize the area, the function may be taken as follows:

$$\begin{aligned} f(x,y,z) &= A^2 \\ &= s(s-x)(s-y)(s-z) \end{aligned}$$

This is subject to constraint the perimeter of rectangle is $g(x,y) = x + y + z = P$.

Recall that, the method of Lagrange multipliers, to find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ (assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x,y,z) = k$):

Find all values of x, y, z and λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = k$.

Evaluate f at all the points (x,y,z) that result from step (1).The largest of these values is the maximum value of f, the smallest is the minimum value of f.

Subject to the constraint,

$$\begin{aligned} g(x,y) &= x + y + z \\ &= P \end{aligned}$$

Use Lagrange multiplier, the two functions can be written as follows:

$f_x = \lambda g_x$ ...... (1)

$f_y = \lambda g_y$ ...... (2)

$f_z = \lambda g_z$ ...... (3)

$x + y + z = P$ ...... (4)

Differentiate the function with respect to x,

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\left[s(s-x)(s-y)(s-z)\right] \\ &= -s(s-y)(s-z) \end{aligned}$$

Differentiate the function with respect to y,

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}\left[s(s-x)(s-y)(s-z)\right] \\ &= -s(s-x)(s-z) \end{aligned}$$

Differentiate the function with respect to z,

$$\begin{aligned} f_z(x,y) &= \frac{\partial}{\partial z}\left[s(s-x)(s-y)(s-z)\right] \\ &= -s(s-x)(s-y) \end{aligned}$$

Differentiate the function with respect to x,

$$\begin{aligned} g_x(x,y) &= \frac{\partial}{\partial x}(x+y+z-P) \\ &= 1 \end{aligned}$$

Differentiate the function with respect to y,

$$\begin{aligned} g_y(x,y) &= \frac{\partial}{\partial y}(x+y+z-P) \\ &= 1 \end{aligned}$$

Differentiate the function with respect to z,

$$\begin{aligned} g_z(x,y) &= \frac{\partial}{\partial z}(x+y+z-P) \\ &= 1 \end{aligned}$$

Use Lagrange multipliers method,

$$f_x = \lambda g_x$$

$$-s(s-y)(s-z) = \lambda \text{ ...... (5)}$$

From the equation (2), to get:

$$f_y = \lambda g_y$$

$$-s(s-x)(s-z) = \lambda \text{ ...... (6)}$$

From the equation (3), to get:

$$f_z = \lambda g_z$$

$$-s(s-x)(s-y) = \lambda \text{ ...... (7)}$$

Divide equation (5) by (6), to get:

$$\frac{s-y}{s-x} = 1$$
$$s-y = s-x$$
$$y = x$$

In the same way divide equation (6) by (7), to obtain:

$$\frac{s-z}{s-y} = 1$$
$$s-z = s-y$$
$$z = y$$

So the three sides are equal i.e. $x = y = z$.

Thus, the triangle with the maximum area with perimeter p is an equilateral triangle.

Answer 29E.

Let the distance between the point (2, 1, -1) and any point (x, y, z) in the point is

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$$

Let $f(x,y,z) = (x-2)^2 + (y-1)^2 + (x+1)^2$ {i.e. d^2}

And the given constraint is

$$g(x,y,z) = x+y-z = 1$$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = 1$

i.e. $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$

And $g(x,y,z) = 1$

i.e.

$$2(x-2) = \lambda \quad \text{------------ (1)}$$
$$2(y-1) = \lambda \quad \text{------------ (2)}$$
$$2(z+1) = -\lambda \quad \text{------------ (3)}$$
$$x+y-z=1 \quad \text{------------ (4)}$$

If x = 0,

From equation (1) $\lambda = -4$

Then from equation (2) and (3)

$y = -1$, $z = 1$

But x = 0, y = -1, z = 1 does not satisfy equation (4)

Now if y = 0,

From equation (2) $\lambda = -2$

Then using this value of λ in equation (1) and (3)

$x = 1$, $z = 0$

Also x = 1, y = 0, z = 0 lies on equation (4)

If z = 0

From equation (3) $\lambda = -2$

Then using this value of λ in equation (1) and (2)

$x = 1$, $y = 0$, $z = 0$

Also from (1); $x = \frac{\lambda}{2} + 2$

From (2) $y = \frac{\lambda}{2} + 1$

From (3) $z = \frac{-\lambda}{2} - 1$

Using these values in equation (3); $\lambda = -2$

Then x = 1, y = 0, z = 0

Now in each case we find the only extreme point of "f" is (1, 0, 0)

Answer 30E.

Let (x, y, z) be the point on the plane $x - y + z = 4$ which is closest to the point $(1, 2, 3)$.

Then the distance between these two points is:

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

Square both sides:

$$\begin{aligned} d^2 &= f(x, y, z) \\ &= (x-1)^2 + (y-2)^2 + (z-3)^2 \end{aligned}$$

And the given constraint is:

$$\begin{aligned} g(x, y, z) &= x - y + z \\ &= 4 \end{aligned}$$

Use Lagrange's method of multipliers to find the values of $x, y \,\&\, z$ for which the distance function is minimum.

Now, by Lagrange's method of multipliers:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

By the above equation:

$$f_x = \lambda g_x,\ f_y = \lambda g_y,\ f_z = \lambda g_z$$

Here f_x, f_y, f_z, g_x, g_y and g_z are partial derivatives.

And

$$g(x, y, z) = 4.$$

So the equations of Lagrange's multipliers give:

$$f_x = \lambda g_x$$
$$2(x-1) = \lambda(1)$$
$$2(x-1) = \lambda$$
$$x = \frac{\lambda}{2} + 1$$

Consider another equation:

$$f_y = \lambda g_y$$
$$2(y-2) = \lambda(-1)$$
$$2(y-2) = -\lambda$$
$$y = -\frac{\lambda}{2} + 2$$

Also,

$$f_z = \lambda g_z$$
$$2(z-3) = \lambda(1)$$
$$2(z-3) = \lambda$$
$$z = \frac{\lambda}{2} + 3$$

From the above equations:

$x = \frac{\lambda}{2} + 1$, $y = -\frac{\lambda}{2} + 2$, and $z = \frac{\lambda}{2} + 3$.

Substitute these values in the equation, $x-y+z=4$:

$$x-y+z=4$$
$$\frac{\lambda}{2}+1-\left(-\frac{\lambda}{2}+2\right)+\frac{\lambda}{2}+3=4$$
$$\frac{\lambda}{2}+1+\frac{\lambda}{2}-2+\frac{\lambda}{2}+3=4$$
$$\frac{3}{2}\lambda=2$$

Therefore the value of $\lambda=\frac{4}{3}$

Then the values of $x, y \,\&\, z$ are:

$x=\frac{5}{3}$, $y=\frac{4}{3}$, $z=\frac{11}{3}$

Hence the extreme point of the function f is only one, that is:

$\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$

Now the physical nature of the problem says that there must be an absolute minimum value of f which makes the two points closest and which has to occur at the extreme point. So it must occur at $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

That is f has minimum value at $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

The distance is:

$$d=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$$
$$=\sqrt{3\left(\frac{2}{3}\right)^2}$$
$$=\frac{2}{3}\sqrt{3}$$

And this is the minimum distance.

Hence the required point is;

$$\boxed{\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)}.$$

Answer 31E.

Let (x, y, z) be the point on the cone $z^2 = x^2 + y^2$ which is closest to the point $(4, 2, 0)$

Then the distance between these two points is:

$$d = \sqrt{(x-4)^2 + (y-2)^2 + (z-0)^2}$$

Let $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$ (i.e. d^2)

And the given constraint is

$$g(x,y,z) = z^2 - x^2 - y^2 = 0$$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = 0$

i.e. $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$

And $g(x,y,z) = 0$

i.e. $2(x-4) = -2x\lambda$ -------------- (1)

$2(y-2) = -2y\lambda$ -------------- (2)

$2z = 2z\lambda$ -------------- (3)

$z^2 - x^2 - y^2 = 0$ -------------- (4)

From equation (3)

We have $\lambda = 1$

Then from (1)

$$2(x-4) = -2x$$

i.e. $4x - 8 = 0 \Rightarrow x = 2$

And from (2)

$$2(y-2) = -2y \Rightarrow y = 1$$

Then from (4)

$$z^2 - 2^2 - 1^2 = 0$$

i.e. $z^2 - 5 = 0$

Or $z = \pm\sqrt{5}$

We find that there are two extreme point of "f"

$$\left(2, 1, \pm\sqrt{5}\right)$$

Now the physical nature of the problem says that there must be an absolute minimum value of "f" which makes the two points closest and which has to occur at the extreme point. So it must occur at $\left(2, 1, \pm\sqrt{5}\right)$

That is "f" has minimum value at $\left(2,1,\pm\sqrt{5}\right)$

This is 10

That is the distance $d=\sqrt{f(x,y,z)}$ will be minimum at $\left(2,1,\pm\sqrt{3}\right)$ and the distance $d=\sqrt{10}$

Hence the required points are $\boxed{\left(2,1,\pm\sqrt{5}\right)}$

Answer 32E.

Consider the following surface:

$$y^2=9+xz$$

The distance from a point (x,y,z) to the point $(0,0,0)$ is given as follows:

$$d=\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

Square both sides, and solve as follows:

$$\begin{aligned}d^2&=(x-0)^2+(y-0)^2+(z-0)^2\\&=x^2+y^2+z^2\end{aligned}$$

The constraint is that the point (x,y,z) lies on the sphere, that is given as follows:

$$g(x,y,z)=y^2-xz=9$$

According to the method of Lagrange multipliers, solve $\nabla f=\lambda\nabla g$, $g=9$ as follows:

$2x=-z\lambda$ ...... (1)

$2y=2\lambda y$ ...... (2)

$2z=-\lambda x$ ...... (3)

$y^2-xz=9$ ...... (4)

The simplest way to solve these equations is to solve for x, y, z in terms of λ from (1) (2) and (3), and then substitute these values into (4).

From (1) obtain the following:

$$2x = -z\lambda$$

$$x = -\frac{z\lambda}{2}$$

From (2) obtain the following:

$$2y = 2\lambda y$$

$1 = \lambda$ or $y = 0$

If $1 = \lambda$ but $y \neq 0$

From (1) obtain the following:

$$2x = -z\lambda$$

$$2x = -z(1)$$

$$x = \frac{-z}{2}$$

From (3) obtain the following:

$$2z = -\lambda x$$

$$2z = -(1)x$$

$$2z = -x$$

$$-2z = x$$

$$x = \frac{-z}{2}$$

$$-2z = \frac{-z}{2}$$

$$4z = z$$

$$3z = 0$$

$$z = 0$$

$$x = \frac{-z}{2}$$

$$x = \frac{0}{2}$$

$$x = 0$$

From (4) obtain the following:

$$y^2 - xz = 9$$

$$y^2 - (0)(0) = 9$$

$$y^2 = \pm 9$$

$$x = \frac{-9}{z}$$

Therefore, the values of $\lambda = 1,\ x,\ y$ and z corresponding to the points (x, y, z) is $(0, -3, 0)$ and $(0, 3, 0)$.

Answer 33E.

Let x, y, z be any three positive numbers.

Sum of the three positive numbers is 100.

That is, $x + y + z = 100.$

Product of the three numbers x, y, z is $f = xyz.$

Consider the function $f(x, y, z) = xyz.$ ...... (1)

To find the maximum and minimum values of the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = x + y + z = 100$ use the Lagrange multipliers.

Differentiate (1) with respect to *x*, get $f_x(x, y, z) = yz.$

Differentiate (1) with respect to *y*, get $f_y(x, y, z) = xz.$

Differentiate (1) with respect to *z*, get $f_z(x, y, z) = xy.$

Differentiate $g(x, y, z)$ with respect to *x*, get $g_x(x, y, z) = 1.$

Differentiate $g(x, y, z)$ with respect to *y*, get $g_y(x, y, z) = 1.$

Differentiate $g(x, y, z)$ with respect to *z*, get $g_z(x, y, z) = 1.$

Solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y, z) = x + y + z = 100$ which can be written as

$$f_x = \lambda g_x$$
$$yz = \lambda(1) \text{ ...... } (2)$$
$$xyz = x\lambda$$
$$f_y = \lambda g_y$$
$$xz = \lambda(1) \text{ ...... } (3)$$
$$xyz = \lambda y$$
$$f_z = \lambda g_z$$
$$xy = \lambda(1) \text{ ...... } (4)$$
$$xyz = \lambda z$$
$$x + y + z = 100 \text{ ...... } (5)$$

From equations (2),(3),(4) notice that $x\lambda = y\lambda = z\lambda$.

That implies $x = y = z$.

Now substitute $y = x, z = x$ in the constraint $x + y + z = 100$ and solve for x.

$$x + y + z = 100$$
$$x + x + x = 100$$
$$3x = 100$$
$$x = \frac{100}{3}$$

Since $x = y = z$, $y = \frac{100}{3}$ and $z = \frac{100}{3}$.

Therefore, the extreme value is $(x, y, z) = \left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$.

Find the function value at the point $\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$:

$$f(x, y, z) = xyz$$
$$f\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right) = \left(\frac{100}{3}\right)\left(\frac{100}{3}\right)\left(\frac{100}{3}\right)$$
$$= \frac{1000000}{27}$$

Hence, the required three positive numbers whose sum is 100 and whose product is a maximum are $(x, y, z) = \boxed{\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)}$ and the maximum product is $\boxed{\frac{1000000}{27}}$.

Answer 34E.

Use Lagrange multipliers to give an alternative solution to find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

Let the three positive numbers be x, y and z. Then

$$f(x,y,z) \equiv x^2 + y^2 + z^2$$

Subject to the constraint

$$g(x,y,z) \equiv x + y + z = 12$$

Use the method of Lagrange multipliers, find the values of x, y, z and λ such that

$$\nabla f = \lambda \nabla g$$

and

$$g(x,y,z) = 12.$$

This gives the equations are,

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ x + y + z &= 12 \end{aligned}$$

Differentiate f partially with respect to x.

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) \\ &= 2x + 0 + 0 \\ &= 2x \end{aligned}$$

Differentiate f partially with respect to y.

$$\begin{aligned} f_y &= \frac{\partial f}{\partial y} \\ &= \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right) \\ &= 0 + 2y + 0 \\ &= 2y \end{aligned}$$

Differentiate f partially with respect to z.

$$\begin{aligned} f_z &= \frac{\partial f}{\partial z} \\ &= \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right) \\ &= 0 + 0 + 2z \\ &= 2z \end{aligned}$$

Differentiate g partially with respect to x.

$$g_x = \frac{\partial g}{\partial x}$$
$$= \frac{\partial}{\partial x}(x+y+z)$$
$$= 1+0+0$$
$$= 1$$

Differentiate g partially with respect to y.

$$g_y = \frac{\partial g}{\partial y}$$
$$= \frac{\partial}{\partial y}(x+y+z)$$
$$= 0+1+0$$
$$= 1$$

Differentiate g partially with respect to z.

$$g_z = \frac{\partial g}{\partial z}$$
$$= \frac{\partial}{\partial z}(x+y+z)$$
$$= 0+0+1$$
$$= 1$$

From the equation $f_x = \lambda g_x$, to get

$$2x = \lambda(1)$$
$$2x = \lambda$$
$$x = \frac{\lambda}{2}$$

From the equation $f_y = \lambda g_y$, to get

$$2y = \lambda(1)$$
$$2y = \lambda$$
$$y = \frac{\lambda}{2}$$

From the equation $f_z = \lambda g_z$, to get

$$2z = \lambda(1)$$
$$2z = \lambda$$
$$z = \frac{\lambda}{2}$$

Substitute the values of x, y and z in $x+y+z=12$.

$$\frac{\lambda}{2}+\frac{\lambda}{2}+\frac{\lambda}{2}=12$$
$$\frac{3\lambda}{2}=12$$
$$\lambda=\frac{24}{3}$$
$$=8$$

The value of x is,

$$x=\frac{\lambda}{2}$$
$$=\frac{8}{2}$$
$$=4$$

The value of y is,

$$y=\frac{\lambda}{2}$$
$$=\frac{8}{2}$$
$$=4$$

The value of z is,

$$z=\frac{\lambda}{2}$$
$$=\frac{8}{2}$$
$$=4$$

Therefore, the three positive numbers are $\boxed{(4,4,4)}$.

Answer 35E.

Let the coordinates of the vertex of the rectangle are (x, y, z)
If the circle is centered at the origin, then dimensions of the box are: $2x, 2y, 2z$
Then the volume of the box is $v = 8xyz$

Take $f(x,y,z)=8xyz$

And the constraint $g(x,y,z)=x^2+y^2+z^2=r^2$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $\quad g(x,y,z) = r^2$

i.e. $\quad f_x = \lambda g_x,\ f_y = \lambda g_y,\ f_z = \lambda g_z$

And $\quad x^2 + y^2 + z^2 = r^2$

i.e.

$$8yz = 2x\lambda \quad \text{------------ (1)}$$

$$8xz = 2y\lambda \quad \text{------------ (2)}$$

$$8xy = 2z\lambda \quad \text{------------- (3)}$$

$$x^2 + y^2 + z^2 = r^2 \quad \text{------------- (4)}$$

From equation (1), (2) and (3)

$$8xyz = 2x^2\lambda$$

$$8xyz = 2y^2\lambda$$

$$8xyz = 2z^2\lambda$$

i.e. $\quad x^2 = y^2 = z^2 \quad \{\lambda \neq 0\}$

(Because if $\lambda = 0$, $x, y, z = 0$ which is not possible because of (4))

Using these values in equation (4)

$$3x^2 = r^2$$

i.e. $\quad x^2 = \dfrac{r^2}{3}$

i.e. $\quad x = \dfrac{r}{\sqrt{3}} \quad \{x > 0\}$

Then $\quad y = \dfrac{r}{\sqrt{3}},\ z = \dfrac{r}{\sqrt{3}}$

Then the only extreme point of "f" is $\left(\dfrac{r}{\sqrt{3}}, \dfrac{r}{\sqrt{3}}, \dfrac{r}{\sqrt{3}}\right)$

Now the physical nature of the problem says that there must be an absolute maximum value of "f" which has to occur at the extreme point. So it must occurs at $\left(\dfrac{r}{\sqrt{3}}, \dfrac{r}{\sqrt{3}}, \dfrac{r}{\sqrt{3}}\right)$

So the volume of the box is maximum when $x = y = z = \dfrac{r}{\sqrt{3}}$

And the volume is; $\quad 8\left(\dfrac{r}{\sqrt{3}}\right)\left(\dfrac{r}{\sqrt{3}}\right)\left(\dfrac{r}{\sqrt{3}}\right) = \dfrac{8r^3}{3\sqrt{3}}$

$$= \boxed{\dfrac{8r^3}{3\sqrt{3}}}$$

Answer 36E.

Consider that the volume of a box $= 1000 \text{ cm}^3$

And the surface area of the box is minimum.

Use Lagrange multipliers to find the dimensions.

Let the dimensions of the box be x, y, z.

Then

Volume of the box $= V$

$= xyz$

$= 1000 \text{ cm}^3$

It is known that, the surface area of the box is $= 2(xy + yz + zx)$

Recall the method of Lagrange multipliers,

To find the maximum or minimum of a function subject to the constraint $g(x, y, z) = k$, $\nabla g \neq 0$

(a) find all the values of *x*, *y*, *z* and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
$$\text{and } g(x, y, z) = k$$

(b) Evaluate the value of *f* at these points. The largest of these values is maximum and the smallest of these values is minimum.

Need to minimize the surface area. So,

$f(x, y, z) = 2(xy + yz + zx)$ ...... (1)

And the surface area is restricted to a volume.

Constraint,

$g(x, y, z) = xyz$

The vector equation $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ in terms of its components is

f are

$f_x=\lambda g_x \quad f_y=\lambda g_y \quad f_z=\lambda g_z \quad g(x,y,z)=1$

Differentiate (1) with respect to *x* partially.

$2(y+z)=\lambda yz$ ...... (2)

Differentiate (1) with respect to *y* partially.

$2(x+z)=\lambda xz$ ...... (3)

Differentiate (1) with respect to *y* partially.

$2(x+y)=\lambda xy$ ...... (4)

And from the equation $g(x,y,z)=1000$

$xyz=1000$ ...... (5)

The values obtained by solving these equations give maximum or minimum values of the given function.

Multiplying equation (2) by *x* and (3) by *y* and (4) by *z* gives

$2x(y+z)=\lambda xyz$

$2y(x+z)=\lambda xyz$

$2z(x+y)=\lambda xyz$

From this three equations,

$xy+xz=yx+yz=zx+zy$ As all are equal to λxyz

Equate either of the two equations and solve for *x*,*y*, and *z*.

First take $xy+xz=yx+yz$. This implies $z=0$ or $x=y$

Next, take $yx+yz=zx+zy$. This implies $x=0$ or $z=y$

Finally Take $xy+xz=zx+zy$. This implies $y=0$ or $x=z$

Therefore, either $x=y=z$ or $x=y=z=0$

But here $x = y = z = 0$ is not applicable as $xyz = 1000$

Use this in (5)

$x^3 = 1000$

$x = \sqrt[3]{1000}$

$= 10$

Therefore, $x = y = z = 10$

Then the only extreme point of "f" is $(10,10,10)$

The physical nature of the problem says that there must be an absolute minimum which must occur at the extreme point.

The function f has the minimum value at $(10,10,10)$.

Minimum surface area of the box $= 2(100 + 100 + 100)$

$= 600 \text{ cm}^2$.

That is the surface area of the box is minimum when $\boxed{x = y = z = 10 \text{ cm}}$

Therefore, the dimensions of the box are $\boxed{10 \text{ cm}, 10 \text{ cm}, \text{ and } 10 \text{ cm}}$.

Answer 37E.

Let the dimensions of the box are; x, y, z

Then the volume of the box is; $v = xyz$

The vertex (x, y, z) lies in the plane $x + 2y + 3z = 6$

Take $f(x,y,z) = xyz$

And the constraint $g(x,y,z) = x + 2y + 3z = 6$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = 6$

i.e. $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$

And $g(x,y,z) = 6$

i.e. $yz = \lambda$ --------- (1)

$xz = 2\lambda$ --------- (2)

$xy = 3\lambda$ --------- (3)

$x + 2y + 3z = 6$ --------- (4)

From equation (1), (2) and (3)

$$xyz = \lambda x = 2\lambda y = 3\lambda z$$

i.e. $x = 2y = 3z \quad \{\lambda \neq 0\}$

(Because if $\lambda = 0$, $x, y, z = 0$ which is not possible because of (4))

i.e. $y = \frac{x}{2} \quad z = \frac{x}{3}$

Using these values in equation (4)

$$x\left(1 + \frac{1}{2}\times 2 + \frac{1}{3}\times 3\right) = 6$$

i.e. $3x = 6$

i.e. $x = 2$

Then $y = 1,\ z = \frac{2}{3}$

Then the only extreme point of "f" is $\left(2, 1, \frac{2}{3}\right)$

The physical nature of the problem says that there must be an absolute maximum value of "f" which has to occur at the extreme point. So the maximum value of "f" occur at x = 2, y = 1 and $z = \frac{2}{3}$

That is the volume of the box is maximum when x = 2, y = 1 and $z = \frac{2}{3}$

And the volume is' $v = (2)(1)\left(\frac{2}{3}\right)$

$$= \boxed{\frac{4}{3}}$$

Answer 38E.

Let the dimensions of the box are ; x, y, z

Then the volume of the box is; $v = xyz$

The surface area of the box is; $2(xy + yz + zx) = 64\ \text{cm}^2$

Take $f(x,y,z) = xyz$

And the constraint; $g(x,y,z) = 2(xy + yz + zx) = 64$

Or $g(x,y,z) = xy + yz + zx = 32$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $\quad g(x,y,z) = 32$

i.e. $\quad f_x = \lambda g_x,\ f_y = \lambda g_y,\ f_z = \lambda g_z$

And $\quad g(x,y,z) = 32$

i.e. $\quad yz = (y+z)\lambda$ --------- (1)

$xz = (x+z)\lambda$ ---------- (2)

$xy = (x+y)\lambda$ ---------- (3)

$xy + yz + zx = 32$ ---------- (4)

From equation (1), (2) and (3)

$$xyz = x(y+z)\lambda = y(x+z)\lambda = z(x+y)\lambda$$

i.e. $\quad xy + xz = yx + yz = zx + zy \qquad \{\lambda \neq 0\}$

(Because if $\lambda = 0$, then $x, y, z = 0$ which is not possible because of (4))

i.e. $\quad x = y = z$

Using this in equation (4)

$$3x^2 = 32$$

i.e. $\quad x^2 = \dfrac{32}{3}$

i.e. $\quad x = 4\sqrt{\dfrac{2}{3}} \qquad (x > 0)$

i.e. $\quad x = y = z = 4\sqrt{\dfrac{2}{3}}$

Then the only extreme point of "f" is

$$\left(4\sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}}\right)$$

Now the physical nature of the problem says that there must be an absolute maximum which must occur at the extreme point.

That is "f" is maximum at $\left(4\sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}}\right)$

That is the volume of the box is maximum when its dimensions are

$$\boxed{x = y = z = 4\sqrt{\frac{2}{3}}}$$

Answer 39E.

Let the dimensions of the box are; x, y, z

Then the volume of the box is; $v = xyz$

And the sum of its edges is; $4(x+y+z) = c$

Take $f(x,y,z) = xyz$

And the constraint $g(x,y,z) = x+y+z = \frac{c}{4}$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = \frac{c}{4}$

i.e. $g_x = \lambda g_x$, $f_y = \lambda g_y$, $g_z = \lambda g_z$

And $g(x,y,z) = \frac{c}{4}$

i.e. $yz = \lambda$ ---------- (1)

$xz = \lambda$ ---------- (2)

$xy = \lambda$ ----------- (3)

$x+y+z = \frac{c}{4}$ ----------- (4)

From equation (1), (2) and (3)

$$xyz = x\lambda = y\lambda = z\lambda$$

i.e. $x = y = z$ $\{\lambda \neq 0\}$

(Because if $\lambda = 0$ then $x, y, z = 0$, which is not possible because of (4))

Using this in equation (4)

$$3x = \frac{c}{4}$$

i.e. $x = \frac{c}{12}$

Then $y = z = \frac{c}{12}$

Then the only extreme point is $\left(\frac{c}{12}, \frac{c}{12}, \frac{c}{12}\right)$

Now the physical nature of the problem says that there must be an absolute maximum value of "f" which has to occur at the extreme point

Thus at $\left(\frac{c}{12}, \frac{c}{12}, \frac{c}{12}\right)$ "f" has maximum value

That is at this point the volume of the box is maximum

Hence the required dimensions of the box are

$$\boxed{x = y = z = \frac{c}{12} \text{ cm}}$$

Answer 40E.

Let the dimensions of the base of the aquarium are x, y and let its height be z

If the cost of making base is five times as that of the walls then the total cost of making aquarium is

$$c = 5xy + 2xz + 2yz$$

And the volume of aquarium is; $xyz = v$ (given)

Take $f(x,y,z) = 5xy + 2xz + 2yz$

And the constraint $g(x,y,z) = xyz = v$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = v$

i.e. $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$

And $g(x,y,z) = v$

i.e. $5y + 2z = \lambda yz$ ----------- (1)

$5x + 2z = \lambda xz$ ----------- (2)

$2x + 2y = \lambda xy$ ----------- (3)

$xyz = v$ ----------- (4)

From equations (1), (2) and (3)

$$\lambda xyz = 5xy + 2xz = 5xy + 2zy = 2xz + 2yz$$

i.e. $x = y = \frac{2}{5}z$

Using these values in equation (4)

$$\frac{4}{25}z^3 = v$$

i.e. $z^3 = \frac{25v}{4}$

i.e. $z = \left(\frac{25v}{4}\right)^{1/3}$

i.e. $z = \frac{5}{2}\left(\frac{2}{5}v\right)^{1/3}$

Then $x = y = \left(\frac{2}{5}v\right)^{1/3}$

Then the only extreme point of "f" is; $\left(\left(\frac{2v}{5}\right)^{1/3}, \left(\frac{2v}{5}\right)^{1/3}, \frac{5}{2}\left(\frac{2v}{5}\right)^{1/3}\right)$

Now the physical nature of the problem says that there must be an absolute minimum value of "f" which has to occur at the extreme point. Then "f" will have minimum value or the cost will be minimum at $\left(\left(\frac{2v}{5}\right)^{1/3}, \left(\frac{2v}{5}\right)^{1/3}, \frac{5}{2}\left(\frac{2v}{5}\right)^{1/3}\right)$

Hence the dimensions of the aquarium are

$$\boxed{x = \left(\frac{2v}{5}\right)^{1/3},\ y = \left(\frac{2v}{5}\right)^{1/3},\ z = \frac{5}{2}\left(\frac{2v}{5}\right)^{1/3}}$$

Answer 41E.

Let the dimensions of the rectangular box are x, y, z. Then the length of the diagonal is

$$L = \sqrt{x^2 + y^2 + z^2}$$

Or $L^2 = x^2 + y^2 + z^2$

The volume of the box is; $v = xyz$

Take $f(x,y,z) = xyz$

And the constant $g(x,y,z) = x^2 + y^2 + z^2 = L^2$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = L^2$

i.e. $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$

And $g(x,y,z) = L^2$

i.e. $yz = \lambda 2x$ ----------- (1)

$xz = \lambda 2y$ ----------- (2)

$xy = \lambda 2z$ ------------ (3)

$x^2 + y^2 + z^2 = L^2$ ------------ (4)

From equation (1), (2) and (3)

$$xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$$

i.e. $x^2 = y^2 = z^2 \quad \{\lambda \neq 0\}$

(Because if $\lambda = 0$, $x, y, z = 0$ which is not possible because of (4))

Using this in equation (4)

$$3x^2 = L^2$$

i.e. $x^2 = \dfrac{L^2}{\sqrt{3}}$

i.e. $x = \dfrac{L}{\sqrt{3}}$ (x > 0)

Then $y = z = \dfrac{L}{\sqrt{3}}$

Then the only extreme point is; $\left(\dfrac{L}{\sqrt{3}}, \dfrac{L}{\sqrt{3}}, \dfrac{L}{\sqrt{3}}\right)$

Now the physical nature of the problem says that there must be an absolute maximum of "f" which has to occur at the extreme point

Therefore "f" will be maximum at $\left(\dfrac{L}{\sqrt{3}}, \dfrac{L}{\sqrt{3}}, \dfrac{L}{\sqrt{3}}\right)$

That is the volume of the box will be maximum when $x = y = z = \dfrac{L}{\sqrt{3}}$

And the maximum volume is; $\boxed{\dfrac{L^3}{3\sqrt{3}}}$

Answer 42E.

Let the length, width, and height of the rectangular box $l, w,$ and h, respectively and the volume V. Then the volume of the box is given by the formula $V = lwh$.
Let us find the extreme values for the volume of a rectangular box.

Since this box must satisfy two constraints. The first constraint is that the total surface area must be 1500 cm^2. The surface consists of six sides. The top and bottom each have surface area lw. The front and back each have surface area hw. The left and right sides each have surface area lh. Let the surface area be A. Then the formula for the surface area is $A = 2lw + 2wh + 2lh$ and the constraint can be written in the form $A = 1500$.

The second constraint is that the total edge length of the box must be 200 cm. For each dimension there are four parallel legs each with the length of the box in that dimension. Depict the edge length with the variable E. The formula for E is $E = 4l + 4w + 4h$ and the constraint can be written as $E = 200$.

To solve this problem using Lagrange multipliers all of the first partials of $f, A,$ and E are needed.
Now differentiating $V, A,$ and E partially on both sides with respect to $l, w,$ and h, we get

$$V_l = wh$$
$$V_w = lh$$
$$V_h = lw$$
$$A_l = 2w + 2h$$
$$A_w = 2l + 2h$$
$$A_h = 2w + 2l$$
$$E_l = 4$$
$$E_w = 4$$
$$E_h = 4$$

Constrained optimization problems can be solved using Lagrange multipliers. The Lagrange multiplier system of equations for the case where there are two constraints can be written as:

$$\nabla f = \lambda \nabla g + \mu h$$
$$g = k$$
$$h = c$$

In this case V plays the role of f, A plays the role of g, and E plays the role of h. Set up the Lagrange system by substituting the formulas and values for the various parts into the Lagrange system of equations.
Since $V_l = \lambda A_l + \mu E_l, V_w = \lambda A_w + \mu E_l$, and $V_h = \lambda A_h + \mu E_h$
Substituting the values, we get

$$wh = \lambda(2w+2h)+4\mu \qquad \text{......(1)}$$
$$lh = \lambda(2l+2h)+4\mu \qquad \text{......(2)}$$
$$lw = \lambda(2w+2l)+4\mu \qquad \text{......(3)}$$
$$1500 = 2lw+2wh+2lh \qquad \text{......(4)}$$
$$200 = 4l+4w+4h \qquad \text{......(5)}$$

Subtract equation (2) from equation (1) and solve.

$$wh - lh = 2\lambda w - 2\lambda l$$
$$h(w-l) = 2\lambda(w-l)$$
$$h = 2\lambda$$

Note that the division in the last step above assumes $w-l$ is not zero. This assumption has not been verified and thus $w-l=0$ is a possible solution. Thus there are two possibilities: $h=2\lambda$ or $w=l$.
Subtract equation (3) from equation (1). The arithmetic and logic is the same as for the case of equation (1) minus equation (2). The result is $l=h$ or $w=2\lambda$. Similarly, equation (2) minus equation (3) yields $h=w$ or $l=2\lambda$.

Consider first the case $w=l$. Substitute into equation (5) and solve.

$$200 = 4l+4l+4h$$
$$50 = 2l+h$$
$$h = 50-2l$$

Substitute $w=l$ and $h=50-2l$ into equation (4). Divide by 2, expand, and collect everything on the left hand side.

$$1500 = 2l^2 + 2l(50-2l) + 2l(50-2l)$$
$$750 = l^2 + 50l - 2l^2 + 50l - 2l^2$$
$$3l^2 - 100l + 750 = 0$$

On comparing the equation $3l^2 - 100l + 750 = 0$ with $ax^2 + bx + c = 0$, we get

$a = 3, b = -100,$ and $c = 750$

Now using quadradic formula,

$$l = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values $a = 3, b = -100,$ and $c = 750$ in the quadratic formula, we obtain

$$\begin{aligned} l &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-100) \pm \sqrt{(-100)^2 - 4(3)(750)}}{2(3)} \\ &= \frac{100 \pm \sqrt{10000 - 9000}}{6} \\ &= \frac{100 \pm \sqrt{1000}}{6} \\ &= \frac{100 \pm 10\sqrt{10}}{6} \\ &= \frac{50 \pm 5\sqrt{10}}{3} \end{aligned}$$

Substitute l into the formulas for w and for h to get

$$\begin{aligned} (l, w, h) &= \left(\frac{50 \pm 5\sqrt{10}}{3}, \frac{50 \pm 5\sqrt{10}}{3}, 50 - 2\left(\frac{50 \pm 5\sqrt{10}}{3} \right) \right) \\ &= \left(\frac{50 \pm 5\sqrt{10}}{3}, \frac{50 \pm 5\sqrt{10}}{3}, \frac{50 \pm 10\sqrt{10}}{3} \right) \end{aligned}$$

Substitute these values into equations (4) and (5) to verify that both roots of the quadratic are valid. They are.

Solve the other two cases either by repeating work similar to that for the case $w=l$ or note that these two cases are the same except that the variables are swapped.
Either method yields the remaining four critical points:

$$(l,w,h)=\left(\frac{50\pm5\sqrt{10}}{3},\frac{50\mp10\sqrt{10}}{3},\frac{50\pm5\sqrt{10}}{3}\right) \text{ and}$$

$$(l,w,h)=\left(\frac{50\mp10\sqrt{10}}{3},\frac{50\pm5\sqrt{10}}{3},\frac{50\pm5\sqrt{10}}{3}\right)$$

Substitute these values into the formula for the volume

$$\begin{aligned}V\left(\frac{50+5\sqrt{10}}{3},\frac{50+5\sqrt{10}}{3},\frac{50-10\sqrt{10}}{3}\right)&=\frac{50+5\sqrt{10}}{3}\cdot\frac{50+5\sqrt{10}}{3}\cdot\frac{50-10\sqrt{10}}{3}\\&=\frac{87500-2500\sqrt{10}}{27}\\&=2947.9373\\&\approx 2947.94\text{ cm}^3\end{aligned}$$

And

$$\begin{aligned}V\left(\frac{50-5\sqrt{10}}{3},\frac{50-5\sqrt{10}}{3},\frac{50+10\sqrt{10}}{3}\right)&=\frac{50-5\sqrt{10}}{3}\cdot\frac{50-5\sqrt{10}}{3}\cdot\frac{50+10\sqrt{10}}{3}\\&=\frac{2500\sqrt{10}+87500}{27}\\&=3533.5442\\&\approx 3533.54\text{ cm}^3\end{aligned}$$

Therefore, the maximum and minimum volume are

$$V=\boxed{3533.54 \text{ and } 2947.94\text{ cm}^3}.$$

Answer 43E.

The given paraboloid is $z=x^2+y^2$
And the given plane is $x+y+2z=2$

The curve of their intersection is obtained by substituting for z from first equation in the second

$$x+y+2\left(x^2+y^2\right)=2$$

i.e. $\quad 2x^2+2y^2+x+y=2$

Now we find the maximum and minimum of the function $f(x,y,z)=z-x^2-y^2$, subject to constraints

$$g(x,y,z)=x+y+2z=2$$
$$h(x,y,z)=2x^2+2y^2+x+y=2$$

By Lagrange's method of multipliers we find all $x, y, z,\ \lambda$ and μ such that

$$\vec{\nabla} f(x,y,z)=\lambda\vec{\nabla} g(x,y,z)+\mu\vec{\nabla} h(x,y,z)$$
$$g(x,y,z)=2$$
$$h(x,y,z)=2$$

i.e. $-2x=\lambda+(4x+1)\mu$ ----------- (1)

$-2y=\lambda+(4y+1)\mu$ ----------- (2)

$1=2\lambda$ ----------- (3)

$x+y+2z=2$ ----------- (4)

$2x^2+2y^2+x+y=2$ ----------- (5)

From equation (3) $\lambda=\frac{1}{2}$

From equation (1) and (2) $x=y$

Using this in equation (5)

$4x^2+2x=2$

i.e. $2x^2+x-1=0$

i.e. $(2x-1)(x+1)=0$

i.e. $x=\frac{1}{2},-1$

i.e. $y=\frac{1}{2},-1$

Using these values in equation (4) we find $z=\frac{1}{2},2$

Then the extreme points of "f" are

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\ (-1,-1,2)$$

Now distance of these points from origin is

$$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{2}\sqrt{3}$$

And $\sqrt{(-1)^2+(-1)^2+(2)^2}=\sqrt{6}$

Therefore the point $\boxed{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$ is the nearest to origin and the point $\boxed{(-1,-1,2)}$ is the farthest from origin

Answer 44E.

Consider the plane

$$4x-3y+8z=5$$

And the cone

$$z^2=x^2+y^2$$

The plane intersects the cone in an ellipse.

Find the equation of this ellipse by finding the intersection of cone and plane.

$$z^2=x^2+y^2$$

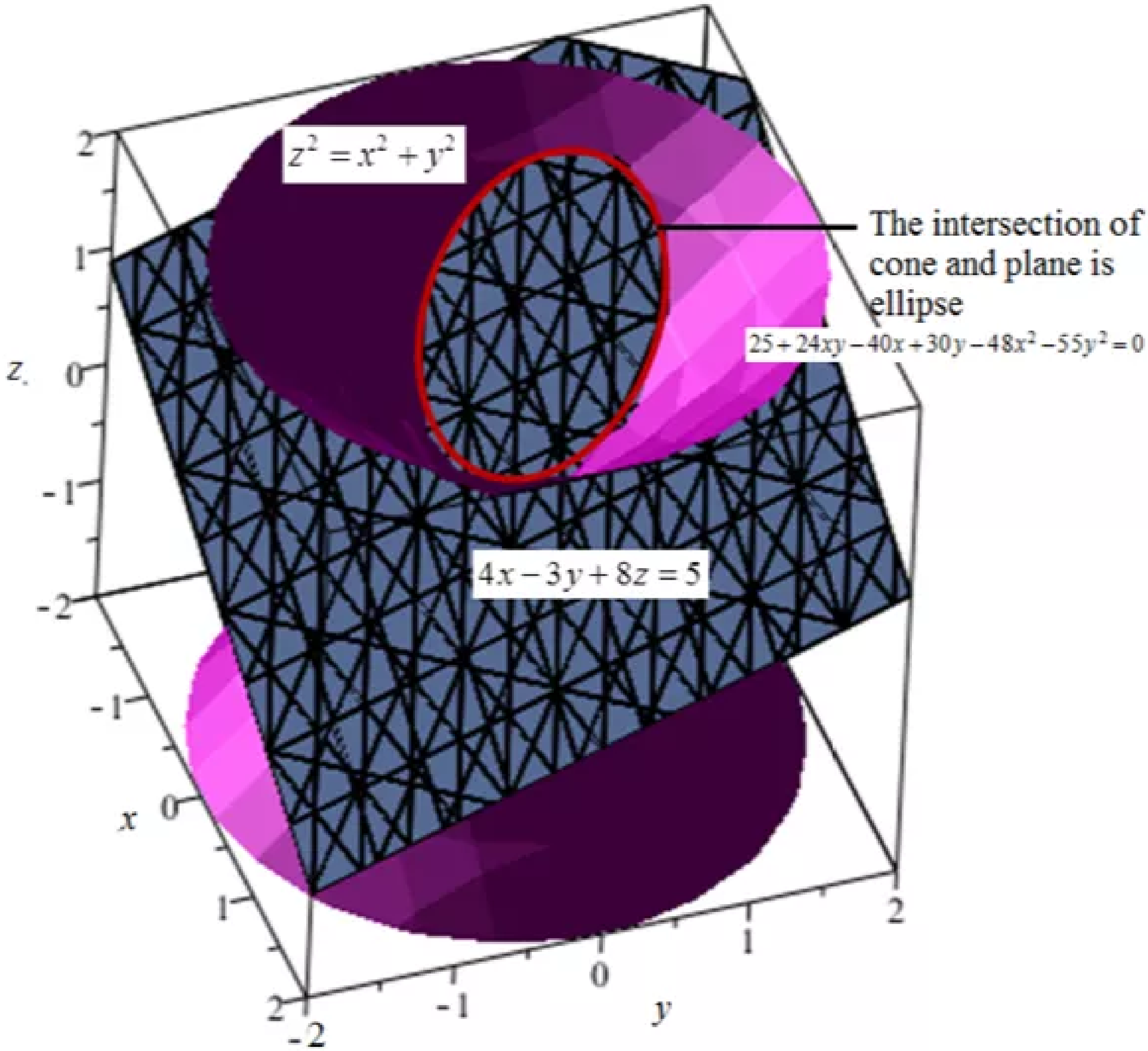
$$\left[\frac{1}{8}(5-4x+3y)\right]^2=x^2+y^2$$ Use $z=\frac{1}{8}(5-4x+3y)$ obtained from plane

$$\frac{1}{64}(5-4x+3y)^2=x^2+y^2$$

$$(5-4x+3y)^2=64x^2+64y^2$$

(a)

Sketch the plane and the cone on the same coordinate axis to find the ellipse.



(b)

Consider the function,

$$f(x,y)=\frac{1}{8}(5-4x+3y)$$

Subject to constraint,

$$g(x,y)=(5-4x+3y)^2-64x^2-64y^2=0$$

Now, use the Lagrange multipliers to find the highest and lowest point on the ellipse.

Recall the method of Lagrange multipliers,

To find the maximum or minimum of a function subject to the constraints $g(x,y,z)=k$,

$$\nabla g \neq 0$$

(a) find all the values of *x*, *y* and λ such that

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z) \text{ and } g(x,y,z)=k$$

(b) Evaluate the value of *f* at these points. The largest of these values is maximum and the smallest of these values is minimum.

The partial derivative of f with respect to *x* is,

$$\begin{aligned}f_x &= \frac{\partial}{\partial x}\left(\frac{1}{8}(5-4x+3y)\right)\\ &= -\frac{4}{8}\\ &= -\frac{1}{2}\end{aligned}$$

The partial derivative of f with respect to *y* is,

$$\begin{aligned}f_y &= \frac{\partial}{\partial y}\left(\frac{1}{8}(5-4x+3y)\right)\\ &= \frac{3}{8}\end{aligned}$$

The partial derivative of g with respect to x is,

$$\begin{aligned}g_x &= \frac{\partial}{\partial x}\left((5-4x+3y)^2-64x^2-64y^2\right)\\ &= 2(5-4x+3y)\cdot(-4)-128x\\ &= -40+32x-24y-128x\\ &= -96x-24y-40\end{aligned}$$

The partial derivative of g with respect to y is,

$$\begin{aligned}g_y &= \frac{\partial}{\partial y}\left((5-4x+3y)^2-64x^2-64y^2\right)\\ &= 2(5-4x+3y)\cdot(3)-128y\\ &= 30-24x+18y-128y\\ &= -24x-110y+30\end{aligned}$$

Using Lagrange's multipliers, solve the equations $\nabla f(x,y)=\lambda\nabla g(x,y)$ and $g(x,y)=0$. This gives the equations

$$f_x=\lambda g_x, f_y=\lambda g_y, \text{ and } g(x,y)=0$$

Substitution yields the equations,

$$f_x=\lambda g_x$$

$$-\frac{1}{2}=\lambda(-96x-24y-40)$$

$$\lambda=-\frac{1}{2(-96x-24y-40)} \quad \text{, ... (1)}$$

$$f_y=\lambda g_y$$

$$\frac{3}{8}=\lambda(-24x-110y+30)$$

$$\lambda=\frac{3}{8(30-24x-110y)} \quad \text{...... (2)}$$

And

$$g(x,y)=0$$

$$(5-4x+3y)^2-64x^2-64y^2=0 \quad \text{...... (3)}$$

Eliminate the variable λ from equations (1) and (2) and then getting equation in x and y .

$$-\frac{1}{2(-40-96x-24y)}=\frac{3}{8(30-24x-110y)}$$

$$8(30-24x-110y)=-6(-40-96x-24y)$$

$$240-192x-880y=240+576x+144y$$

$$768x+1024y=0$$

$$768x=-1024y$$

$$x=-\frac{1024}{768}y$$

$$x=-\frac{32\times 32}{8\times 8\times 12}y$$

$$=-\frac{8\times 4\times 8\times 4}{8\times 8\times 4\times 3}y$$

$$=-\frac{4\times 4}{4\times 3}y$$

$$=-\frac{4}{3}y$$

$$y=-\frac{3}{4}x$$

Substitute *y* value in equation (3) to obtain the values of *x* and *y*.

$$(5-4x+3y)^2-64x^2-64y^2=0$$

$$\left(5-4x+3\left(-\frac{3}{4}x\right)\right)^2-64x^2-64\left(-\frac{3}{4}x\right)^2=0$$

$$\left(5-4x-\frac{9}{4}x\right)^2-64x^2-64\times\frac{9x^2}{16}=0$$

$$\frac{(20-16x-9x)^2}{16}-64x^2\left(1+\frac{9}{16}\right)=0$$

$$(20-25x)^2-64\times 25x^2=0$$

$$(20-25x)^2=64\times 25x^2$$

$$(20-25x)^2=1600x^2$$

$$(20-25x)^2=(40x)^2$$

$$20-25x=40x \text{ or } 20-25x=-40x$$

$$20=65x \quad \text{or } 20=-15x$$

$$x=\frac{20}{65} \quad \text{or } x=-\frac{4}{3}$$

$$x=\frac{4}{13} \quad \text{or } x=-\frac{4}{3}$$

To find the *y* values, substitute the *x* values in $y=-\frac{3}{4}x$.

$$y=-\frac{3}{4}\times\frac{4}{13} \qquad \text{Substiute } x=\frac{4}{13}$$

$$=-\frac{3}{13}$$

$$y=-\frac{3}{4}\times\left(-\frac{4}{3}\right) \qquad \text{Substiute} x=-\frac{4}{3}$$

$$=1$$

Therefore, the function *f* has a possible extreme values at the points $\left(\frac{4}{13},-\frac{3}{13}\right)$ and $\left(-\frac{4}{3},1\right)$.

Evaluate *f* at these two points:

For the point $\left(\frac{4}{13},-\frac{3}{13}\right)$,

$$f\left(\frac{4}{13},-\frac{3}{13}\right)=\frac{1}{8}\left(5-4\left(\frac{4}{13}\right)+3\left(-\frac{3}{13}\right)\right)$$

Use $f(x,y)=\frac{1}{8}(5-4x+3y)$

$$=\frac{1}{8}\left(\frac{65-16-9}{13}\right)$$

$$=\frac{1}{8}\left(\frac{40}{13}\right)$$

$$=\frac{5}{13}$$

For the point $\left(-\frac{4}{3},1\right)$,

$$f\left(-\frac{4}{3},1\right)=\frac{1}{8}\left(5-4\left(-\frac{4}{3}\right)+3(1)\right)$$

Use $f(x,y)=\frac{1}{8}(5-4x+3y)$

$$=\frac{1}{8}\left(5+\frac{16}{3}+3\right)$$

$$=\frac{1}{8}\left(8+\frac{16}{3}\right)$$

$$=\frac{1}{8}\left(\frac{24+16}{3}\right)$$

$$=\frac{5}{3}$$

From the above two values, the maximum value of the function occurs at $\boxed{\left(-\frac{4}{3},1\right)}$ and minimum value of *f* occurs at $\boxed{\left(\frac{4}{13},-\frac{3}{13}\right)}$.

Therefore, the highest point on the ellipse is $\boxed{\left(-\frac{4}{3},1,\frac{5}{3}\right)}$ and lowest point on the ellipse is $\boxed{\left(\frac{4}{13},-\frac{3}{13},\frac{5}{13}\right)}$.

Answer 45E.

Assume that f and g have continuous first partial derivatives and that f has an extremum at point $P(x_0, y_0, z_0)$ on the smooth constraint curve $g(x,y,z)=c_2$. If $\nabla g(x,y,z)\neq 0$ and then there exist a real number λ such that $\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)$.

Consider the function $f(x,y,z)=ye^{x-z}$ subject to the following constraints $g(x,y,z)=9x^2+4y^2+36z^2=36$ and $h(x,y,z)=xy+yz=1$.

From the second constraint $xy+yz=1$, write y in terms of x and z as $y=\frac{1}{x+z}$.

Substitute the value for y in $f(x,y,z)$ and the constraint $g(x,y,z)$. To get

$$f(x,z)=\frac{e^{x-z}}{x+z} \text{ and } g(x,z)=9x^2+\frac{4}{(x+z)^2}+36z^2=36$$

From the Lagrange multiplier equation, $\nabla f(x,z)=\lambda\nabla g(x,z)$.

Compare the coordinates, to get $f_x=\lambda g_x$ and $f_y=\lambda g_y$

Thus $f_x=\lambda g_x$ gives $\frac{e^{x-z}(x+z-1)}{(x+z)^2}=\lambda\left[18x-\frac{8}{(x+z)^3}\right]$ ...... (1)

And $f_y=\lambda g_y$ gives $\frac{e^{x-z}(x+z+1)}{(x+z)^2}=\lambda\left[72z-\frac{8}{(x+z)^3}\right]$ ...... (2)

Divide equation (1) by equation (2), to get

$$\frac{x+z-1}{x+z+1}=\frac{18x(x+z)^3-8}{72z(x+z)^3-8} \quad \text{...... (3)}$$

Solve the equation (3) and the constraint equation $g(x,z)=9x^2+\frac{4}{(x+z)^2}+36z^2=36.$

Use CAS (computer algebra system) to solve the system of equations.

There are four solutions:

$$x\approx 0.233729 \quad z\approx -0.799567$$
$$x\approx -1.94138 \quad z\approx -0.181993$$
$$x\approx 1.06774 \quad z\approx 0.827063$$
$$x\approx 1.36679 \quad z\approx -0.590441$$

And the corresponding values of the function $f(x,z)$ are:

$$f(0.233729,-0.799567)\approx -4.96664$$
$$f(-1.94138,-0.181993)\approx -0.0810740$$
$$f(1.06774,0.827063)\approx 0.671368$$
$$f(1.36679,-0.590441)\approx 9.11922$$

Hence, the maximum value of f **is** 9.11922 **and minimum value** f **is** -4.96664 .

Answer 46E.

$$f(x,y,z)=x+y+z,\ g(x,y,z)=x^2-y^2-z=0,\ h(x,y,z)=x^2+z^2=4$$

$$\nabla f=\lambda\nabla g+\mu\nabla h\Rightarrow\langle 1,1,1\rangle=\lambda\left(2x,\ -2y,\ -1\right)+\mu\left(2x,\ 0,\ 2z\right),\ \text{so } 1=2\lambda x\cdot$$

By solving these 5 equations simultaneously for x, y, z, λ, and μ, we get 4 real-valued solutions:

$$x\approx -1.652878,\ y\approx -1.964194,\ z\approx -1.126052,\ \lambda\approx 0.254557,\ \mu\approx -0.557($$

$$x\approx -1.502800,\ y\approx 0.968872,\ z\approx 1.319694,\ \lambda\approx -0.516064,\ \mu\approx 0.183352$$

$$x\approx -0.992513,\ y\approx 1.649677,\ z\approx -1.736352,\ \lambda\approx -0.303090,\ \mu\approx -0.200($$

$$x\approx 1.895178,\ y\approx 1.718347,\ z\approx 0.638984,\ \lambda\approx -0.290977,\ \mu\approx 0.554805$$

Substituting these values into f gives

$$f(-1.652878,\ -1.964194,\ -1.126052)\approx -4.7431,\ f(-1.502800,\ 0.968872,\ 1.$$

$$f(-0.992513,\ 1.649677,\ -1.736352)\approx -1.0792,\ f(1.895178,\ 1.718347,\ 0.6389$$

Therefore, the maximum is approximately 4.2525, and the minimum is approximately -4.7431.

Answer 47E.

(a) Consider the function:

$$f(x_1,x_2,\dots,x_n)=\sqrt[n]{x_1x_2\cdots x_n}.$$

Where $x_1,x_2,\dots,x_n$ are positive numbers, and $x_1+x_n+\dots+x_n=c$, where c is a constant.

Let the constraint is $g(x_1,x_2,\dots,x_n)=x_1+x_n+\dots+x_n=c$

Take $h(x_1,x_2,\dots,x_n)=x_1x_2\cdots x_n$ {i.e. f^n}.

By Lagrange's method of multipliers, we find all $x_1,x_2,\dots,x_n$ and λ such that

$\nabla h(x_1,x_2,\dots,x_n)=\lambda\nabla g(x_1,x_2,\dots,x_n)$ And $g(x_1,x_2,\dots,x_n)=c$

By applying the Lagrange's method of multipliers, we get

$$\begin{aligned}\nabla h(x_1,x_2,\dots,x_n)&=\lambda\nabla g(x_1,x_2,\dots,x_n)\\ \frac{\partial}{\partial x_1}(x_1x_2\dots x_n)&=\lambda\frac{\partial}{\partial x_1}(x_1+x_2+x_3+\dots+x_n)\\ x_2x_3\dots x_n&=\lambda(1)\\ x_2x_3\dots x_n&=\lambda\end{aligned}$$

Multiply the equation, $x_2x_3\dots x_n=\lambda$ with x_1 on both sides, we get

$$x_1x_2x_3\dots x_n=x_1\lambda \qquad \dots\dots (1)$$

And

$$\begin{aligned}\nabla h(x_1,x_2,\dots,x_n)&=\lambda\nabla g(x_1,x_2,\dots,x_n)\\ \frac{\partial}{\partial x_2}(x_1x_2\dots x_n)&=\lambda\frac{\partial}{\partial x_2}(x_1+x_2+x_3+\dots+x_n)\\ x_1x_3\dots x_n&=\lambda(1)\\ x_1x_3\dots x_n&=\lambda\end{aligned}$$

Multiply the equation, $x_1x_3\dots x_n=\lambda$ with x_2 on both sides, we get

$$x_1x_2\dots x_n=x_2\lambda \qquad \dots\dots (2).$$

And so on,

$$\nabla h(x_1,x_2,...,x_n)=\lambda\nabla g(x_1,x_2,...,x_n)$$
$$\frac{\partial}{\partial x_n}(x_1x_2...x_n)=\lambda\frac{\partial}{\partial x_n}(x_1+x_2+x_3+....+x_n)$$
$$x_1x_2...x_{n-1}=\lambda(1)$$
$$x_1x_2...x_{n-1}=\lambda$$

Multiply the equation, $x_1x_2...x_{n-1}=\lambda$ with x_n on both sides, we get

$$x_1x_2...x_{n-1}x_n=x_n\lambda \qquad ......\ (n-1)$$

If $c\neq 0$ add all the above equations, we get

$$n(x_1x_2...x_n)=\lambda(x_1+x_2+........+x_n).$$

We know that $x_1+x_2+...+x_n=c \qquad ......\ (n)$

Using the above equation, we get

$$n(x_1x_2...x_n)=\lambda c$$
$$x_1x_2...x_n=\frac{\lambda c}{n}$$

From the equation (1), and the equation, $x_1x_2...x_n=\frac{\lambda c}{n}$, we get

$$\frac{\lambda c}{n}=\lambda x_1$$
$$x_1=\frac{c}{n}$$

Similarly, we can find the values, $x_2=\frac{c}{n}$, $x_3=\frac{c}{n}$, and so on $x_n=\frac{c}{n}$.

Suppose, $c=0$

Since $x_1,x_2,..........,x_n$ are positive numbers then the values, $x_1=x_2=.........=x_n=0$.

So the extreme points of h are

And so on,

$$\nabla h(x_1, x_2, ..., x_n) = \lambda \nabla g(x_1, x_2, ..., x_n)$$

$$\frac{\partial}{\partial x_n}(x_1 x_2 ... x_n) = \lambda \frac{\partial}{\partial x_n}(x_1 + x_2 + x_3 + ... + x_n)$$

$$x_1 x_2 ... x_{n-1} = \lambda(1)$$

$$x_1 x_2 ... x_{n-1} = \lambda$$

Multiply the equation, $x_1 x_2 ... x_{n-1} = \lambda$ with x_n on both sides, we get

$$x_1 x_2 ... x_{n-1} x_n = x_n \lambda \qquad ...... (n-1)$$

If $c \neq 0$ add all the above equations, we get

$$n(x_1 x_2 ... x_n) = \lambda(x_1 + x_2 + ... + x_n).$$

We know that $x_1 + x_2 + ... + x_n = c \qquad ...... (n)$

Using the above equation, we get

$$n(x_1 x_2 ... x_n) = \lambda c$$

$$x_1 x_2 ... x_n = \frac{\lambda c}{n}$$

From the equation (1), and the equation, $x_1 x_2 ... x_n = \frac{\lambda c}{n}$, we get

$$\frac{\lambda c}{n} = \lambda x_1$$

$$x_1 = \frac{c}{n}$$

Similarly, we can find the values, $x_2 = \frac{c}{n}$, $x_3 = \frac{c}{n}$, and so on $x_n = \frac{c}{n}$.

Suppose, $c = 0$

Since $x_1, x_2, ..., x_n$ are positive numbers then the values, $x_1 = x_2 = ... = x_n = 0$.

So the extreme points of h are

$$\left(\frac{c}{n},\frac{c}{n},\dots,\frac{c}{n}\right)=(0,0,\dots,0)$$

Now

$$h\left(\frac{c}{n},\frac{c}{n},\dots,\frac{c}{n}\right)=\left(\frac{c}{n}\right)\left(\frac{c}{n}\right)\cdots\left(\frac{c}{n}\right)$$

$$h\left(\frac{c}{n},\frac{c}{n},\dots,\frac{c}{n}\right)=\left(\frac{c}{n}\right)^n$$

$$h(0,0,\dots\dots\dots,0)=0^n$$
$$=0$$

Then the maximum value of h is $\left(\frac{c}{n}\right)^n$.

Now if h is maximum then " f " is also maximum, and then the maximum value of " f " is

$$\left[\left(\frac{c}{n}\right)^n\right]^{1/n}=\boxed{\frac{c}{n}}.$$

(b) From part (a), we find the maximum value of the function $f(x_1,x_2,\dots,x_n)=\sqrt[n]{x_1x_2\dots.x_n}$ is $\frac{c}{n}$

Where $c=x_1+x_2+\dots+x_n$

That is for all $x_1,x_2,\dots,x_n$

$$\sqrt[n]{x_1x_2\dots x_n}\le\frac{c}{n}$$

$$\sqrt[n]{x_1x_2\dots x_n}\le\frac{x_1+x_2+\dots+x_n}{n}$$

But if $x_1=x_2=\dots=x_n$

Then

$$\sqrt[n]{x_1x_2\dots x_n}=\sqrt[n]{x_1x_1\dots x_1}$$
$$=(x_1x_1\dots x_1)^{\frac{1}{n}}$$
$$=\left((x_1)^n\right)^{\frac{1}{n}}$$
$$=x_1$$

And also

$$\frac{x_1+x_2+\dots\dots\dots x_n}{n}=\frac{x_1+x_1+\dots\dots\dots x_1}{n}$$
$$=\frac{nx_1}{n}$$
$$=x_1$$

That is if $x_1=x_2=\dots\dots\dots=x_n$ only the geometric mean and arithmetic means are equal.

That is $\sqrt[n]{x_1x_2\dots\dots\dots x_n}=\dfrac{x_1+x_2+\dots\dots+x_n}{n}$.

Answer 48E.

Consider the following function:

$$f(x_1,x_2,...,x_n,y_1,y_2,...,y_n)=\sum x_i y_i$$

Here, the function is subject to the following constraints:

$$\sum x_i^2=1$$

$$\sum y_i^2=1$$

Use the following Lagrange multipliers method, to find the maximum or minimum of a function subject to the constraints $g(x,y)=k,\ \ \nabla g\neq 0,\ \ h(x,y)=c,\nabla h\neq 0$:

(a) Find all the values of *x*, *y*, *z* and λ such that

$$\nabla f(x,y)=\lambda\nabla g(x,y)+\mu\nabla h(x,y)$$
and $g(x,y)=k, h(x,y)=c$

(b) Evaluate the value of *f* at these points. The largest of these values is the maximum, and the smallest of these values is the minimum of a function.

Compare the given constraints with $g(x,y)=k,$ and $h(x,y)=c,$ to get the following:

$$g(x,y)=\sum x_i^2$$
$$h(x,y)=\sum y_i^2$$

Then, the components of $\nabla f, \nabla g$ and ∇h are as follows:

$$\nabla f=\langle y_1,y_2,...,y_n,x_1,x_2,...,x_n\rangle$$
$$\nabla g=\langle 2x_1,2x_2,...,2x_n,0,0,...,0\rangle$$
$$\nabla h=\langle 0,0,...,0,2y_1,2y_2,...,2y_n\rangle$$

The Lagrange multiplier system of equations for the case where, there are two constraints can be written as follows:

$$\nabla f = \lambda \nabla g + \mu \nabla h$$
$$g = k$$
$$h = c$$

The vector equation $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ in terms of its components is:

$$f_x = \lambda g_x + \mu h_x, f_y = \lambda g_y + \mu h_y \quad , f_z = \lambda g_z + \mu h_z$$

Set up the Lagrange system by substituting the formulas, and values for the various parts into the Lagrange system of equations.

Substitution yields the following equations:

$$y_i = 2\lambda x_i, 1 \le i \le n \ \ldots\ldots \quad (2)$$

$$x_i = 2\mu y_i, 1 \le i \le n \ \ldots\ldots \quad (3)$$

$$\sum x_i^2 = 1 \ \ldots\ldots \quad (4)$$

$$\sum y_i^2 = 1 \ \ldots\ldots \quad (5)$$

Substitute equation (2) $y_i = 2\lambda x_i$ in equation (5) $\sum y_i^2 = 1$, to get the following:

$$\sum (2\lambda x_i)^2 = 1$$
$$\sum 4\lambda^2 x_i^2 = 1$$
$$4\lambda^2 \sum x_i^2 = 1$$

$4\lambda^2 = 1$ Use (4)

$$\lambda^2 = \frac{1}{4}$$
$$\lambda = \pm \frac{1}{2}$$

Find y_i when, $\lambda = \frac{1}{2}$:

Substitute $\lambda = \frac{1}{2}$ in (2), to get the following:

$y_i = 2\left(\frac{1}{2}\right)x_i, 1 \le i \le n$ Plug in $\frac{1}{2}$ for λ

$= x_i$

Find y_i when, $\lambda = -\frac{1}{2}$:

Substitute $\lambda = -\frac{1}{2}$ in (2), to get the following:

$y_i = 2\left(-\frac{1}{2}\right)x_i, 1 \le i \le n$ Plug in $-\frac{1}{2}$ for λ

$= -x_i$

Evaluate $f(x, y)$ when, $y_i = x_i$ as follows:

$$
\begin{aligned}
f(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) &= \sum x_i x_i \\
&= \sum x_i^2 \quad \text{Use constraint } \sum x_i^2 = 1 \\
&= 1
\end{aligned}
$$

Evaluate $f(x, y)$ when, $y_i = -x_i$ as follows:

$$
\begin{aligned}
f(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) &= \sum x_i(-x_i) \\
&= -\sum x_i^2 \quad \text{Use constraint } \sum x_i^2 = 1 \\
&= -1
\end{aligned}
$$

Substitute equation (2) $x_i = 2\mu x y_i$ in equation (5) $\sum x_i^2 = 1$.

$$\sum (2\mu y_i)^2 = 1$$

$$\sum 4m^2 y_i^2 = 1$$

$$4\mu^2 \sum y_i^2 = 1$$

$4\mu^2 = 1$ Use (5)

$$\mu^2 = \frac{1}{4}$$

$$\mu = \pm\frac{1}{2}$$

Find x_i when, $\mu = \frac{1}{2}$:

Substitute $\mu = \frac{1}{2}$ in (2), to get the following:

$$x_i = 2\left(\frac{1}{2}\right)y_i, 1 \le i \le n \text{ Plug in } \frac{1}{2} \text{ for } \mu$$

$$= y_i$$

Find x_i when, $\mu = -\frac{1}{2}$:

Substitute $\mu = -\frac{1}{2}$ in (2), to get the following:

$$x_i = 2\left(-\frac{1}{2}\right)y_i, 1 \le i \le n \text{ Plug in } -\frac{1}{2} \text{ for } \mu$$

$$= -y_i$$

Evaluate $f(x,y)$ when, $x_i = y_i$ as follows:

$$\begin{aligned} f(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) &= \sum y_i y_i \\ &= \sum y_i^2 \quad \text{Use constraint } \sum y_i^2 = 1 \\ &= 1 \end{aligned}$$

Evaluate $f(x,y)$ when, $x_i = -y_i$ as follows:

$$\begin{aligned} f(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) &= \sum (-y_i) y_i \\ &= -\sum y_i^2 \quad \text{Use constraint } \sum y_i^2 = 1 \\ &= -1 \end{aligned}$$

So, for all the values of λ, μ, the function $f(x,y)$ takes values -1, and 1.

Therefore, the maximum value of the function f subject to given constraints is $\boxed{1}$.

(b)

Consider the following conditions:

$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}}$$

$$y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$

Assume that $\sum_{j=1}^{n} a_j^2 \neq 0$ and $\sum_{j=1}^{n} b_j^2 \neq 0.$

Here, the function attains the same maximum with $x_i = \frac{a_i}{\sqrt{\sum a_j^2}}, y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$.

So, the summation $\sum x_i y_i, 1 \leq i \leq n$ is less than or equal to 1 because, it can attain a maximum value 1.

This above analysis can be mathematically written as follows:

$$\sum x_i y_i \leq 1$$

$$\sum \frac{a_i}{\sqrt{\sum a_j^2}} \cdot \frac{b_i}{\sqrt{\sum b_j^2}} \leq 1 \qquad \text{Substitute } x_i = \frac{a_i}{\sqrt{\sum a_j^2}}, \text{ and } y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$

$$\frac{1}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \sum a_i b_i \leq 1$$

$$\sum a_i b_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$$

Therefore, $\boxed{\sum a_i b_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}}$.