VECTOR CALCULUS

1) grad ϕ OR $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\phi(x,y,z) = c$ (scalar)

2) grad $\vec{r} = \frac{\vec{r}}{r}$ Short-cut $\nabla f(\overset{r}{x}) = f'(x)\frac{\vec{r}}{r}$

3) Tangent $\vec{r}(t)$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

4) Normal

$$N = \frac{\nabla\cdot\phi}{|\nabla\cdot\phi|}$$

- Gradient of constant is zero.
- $\nabla(f \pm g) = \nabla f \pm \nabla g$
- $\text{grad}(fg) = f\,\text{grad}\,g + g\,\text{grad}\,f$
- $\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}$

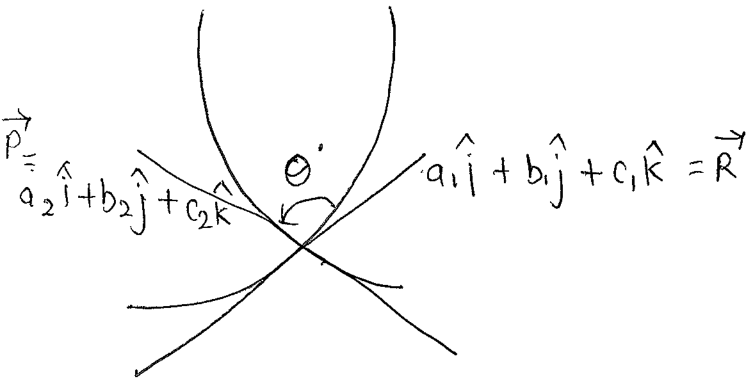
direction derivative

5) $D.D = \nabla\cdot\phi\, e$

$$e = \frac{\vec{a}}{|a|}$$

Angle between two curves

- It is the angle between tangent drawn at this curve



Then $\vec{R}$ ~~be~~ and $\vec{P}$ be any two tangent vectors and θ is the angle between

$$\cos\theta = \frac{P\cdot R}{|\vec{P}|\,|\vec{R}|}$$; $P\cdot R = a_1a_2 + b_1b_2 + c_1c_2$

NOTE:

If the two curves cut each other orthogonally then $P \cdot R = 0$

Angle between two surface

- Angle betⁿ 2 surface is angle betⁿ normal drawn at that point of intersection. If f and g are any two surface and θ is the angle between them then $\boxed{\cos\theta = \dfrac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}}$

Q:- Find unit normal vector to a surface

$$x^3 + y^3 + 3xyz = 3 \quad \text{at } (1, 2, -1)$$

$$x^3 + y^3 + 3xyz - 3 = 0$$

unit vector

$$N = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\nabla\phi = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

$$\nabla\phi_{(1,2,-1)} = [3 + 3(2)(-1)]\hat{i} + [3(4) + 3(1)(-1)]\hat{j} + [3(1)(2)]\hat{k}$$

$$= (3-6)\hat{i} + (12-3)\hat{j} + 6\hat{k}$$

$$\nabla\phi_{(1,2,-1)} = -3\hat{i} + 9\hat{j} + 6\hat{k}$$

$$N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{9+81+36}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{126}}.$$

Q:- Find unit normal vector $x^2z + 2xy = 4$ at $(2,-2,3)$

$$N = \frac{\nabla\phi}{\nabla\phi}$$

$$\nabla\phi = (2xz + 2y)\hat{i} + (2x)\hat{j} + (x^2)\hat{k}$$

$$\nabla\phi_{(2,-2,3)} = [12 + (-4)]\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\nabla\phi_{(2,-2,3)} = 8\hat{i} + 4\hat{j} + 4\hat{k}$$

$$|\nabla\phi| = \sqrt{16+16+64}$$

$$N = \frac{8\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{96}}$$

$$N = \frac{\cancel{4}\,(2\hat{i} + \hat{j} + \hat{k})}{\cancel{4}\,\sqrt{6}}$$

$$N = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

Q:- Find the max^m value of D.D. $\phi(x^2yz^3)$ at $(2,1,-1)$

$$D.D. = |\nabla\phi|$$

$$\nabla\phi = (2xyz^3)\hat{i} + (x^2z^3)\hat{j} + (x^2y\cdot 3z^2)\hat{k}$$

$$= 2(2)(1)(-1) + 4\cdot(-1) + 4\cdot 1\cdot 3(1)$$

$$\nabla\phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$$

$$|\nabla\phi| = \sqrt{4^2+4^2+12^2} = \sqrt{16+16+144} = \sqrt{176} = 4\sqrt{11}$$

Q:- Magnitude of gradient

$$u = \frac{x^2}{2} + \frac{y^2}{3} \quad \text{at } (1,3)$$

$$\nabla u = \left(\frac{2x}{2}\right)\hat{i} + \left(\frac{2y}{3}\right)\hat{j}$$

$$= 1\hat{i} + 2\hat{j}$$

$$|\nabla u| = \sqrt{5}$$

Q:- Find the D.D. of a function $f = xy + yz + zx$ in direction of $\hat{i} + 2\hat{j} + 2\hat{k}$. at $(1,2,0)$

$$D.D. = \nabla\phi \cdot e \quad , \quad e = \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla\phi = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k} = \cancel{2\hat{i} + \hat{j} + 3\hat{k}}$$

$$e = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

$$D.D. = \nabla\phi \cdot e$$

$$= \cancel{(2\hat{i} + \hat{j} + 3\hat{k})} \; \frac{(\hat{i} + 2\hat{j} + 2\hat{k})}{3} \cdot \left((y+z)\hat{i} + (x+z)\hat{j} + (y+z)\hat{k}\right)$$

$$D.D. = \cancel{\frac{2+1+6}{3}} = \frac{y + z + 2(x+z) + 2(y+z)}{3}$$

$$= \frac{2 + 2 + 2(2)}{3} = \frac{10}{3}$$

Q:- Find D.D. of $f = 2xy + z^2$ at $(1,-1,3)$ in the direction of $\hat{i} + 2\hat{j} + 3\hat{k}$

$$\nabla f = (2y)\hat{i} + (2x)\hat{j} + (2z)\hat{k}$$

$$D.D. = \nabla \cancel{\phi} f \cdot e$$

$$= \frac{2y + 4x + 6z}{\sqrt{1+4+9}}$$

$$= \frac{2(-1) + 4(1) + 6(3)}{\sqrt{14}}$$

$$= \frac{-2 + 4 + 18}{\sqrt{14}}$$

$$D.D. = \frac{20}{\sqrt{14}}$$

Q:- Find D.D. of $f = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in direction of $\overline{PQ}$ where $Q(5,0,4)$

$$\overline{PQ} = \overline{Q} - \overline{P}$$

$$\overline{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\nabla f = (2x)\hat{i} + (-2y)\hat{j} + (4z)\hat{k}$$

$$D.D. = \nabla f \cdot e$$

$$= \frac{8x + 4y + 4z}{\sqrt{16+4+1}} = \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

Q: Find direction derivative (D.D.) of $f = xy^2 + yz^2 + zx^2$ at $(1,1,1)$ along the tangent $x = t, y = t^2, z = t^3$

$$\nabla\phi = (y^2 + 2xz)\hat{i} + (2xy + z^2)\hat{j} + (2yz + x^2)\hat{k}$$

$$D.D. = \nabla\phi \cdot e$$

$$= \frac{(y^2+2xz)\hat{i} + (2xy+z^2)\hat{j} + (2yz+x^2)\hat{k} \cdot (1\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}}$$

$$= \frac{y^2 + 2xz + 2(2xy+z^2) + 3(2yz+x^2)}{\sqrt{14}}$$

$$= \frac{1+2+2(2+1)+3(2+1)}{\sqrt{14}} = \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

Q: Find D.D. of field $u(x,y,z) = x^2 - 3yz$ in the direction of vector $\hat{i} + \hat{j} - 2\hat{k}$ at point $(2,-1,4)$ is ____

$$\nabla\phi = (2x)\hat{i} - (3z)\hat{j} + (-3y)\hat{k}$$

$$D.D. = \nabla\phi \cdot e$$

$$= (2x)\hat{i} - (3z)\hat{j} + (-3y)\hat{k} \cdot \frac{(\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{1+1+4}}$$

$$= \frac{2x - 3z + 6y}{\sqrt{6}} = \frac{4-12-6}{\sqrt{6}} = \frac{-14}{\sqrt{6}} = -5.72$$

Q:- The angle betⁿ the surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at $(2,-1,2)$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 4\hat{i} + (-2)\hat{j} + 4\hat{k}$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} - \hat{k} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|} = \frac{4x^2 + 4y^2 + (-2z)}{\sqrt{16+4+16}\cdot\sqrt{16+4+1}}$$

$$\cos\theta = \frac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \frac{16}{6\cdot\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

Q:- Find the cons. a and b so that surface $ax^2-byz=(a+2)x$ and $4x^2y+z^3=4$. intersect orthogonally at $(1,-1,2)$

$$ax^2 - byz - (a+2)x = 0$$

$$\nabla f = [2ax-(a+2)]\hat{i} + (-bz)\hat{j} + (-by)\hat{k} = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$$

$$\nabla g = (8xy)\hat{i} + (4x^2)\hat{j} + (3z^2)\hat{k} = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

$$\nabla f \cdot \nabla g = 0$$

$$\cancel{-8((2ax-(a+2)x)}\quad (a-2)(-8) + 4(-2b) + 12b = 0$$

$$-8a + 16 - 8b + 12b$$

$$-8a + 4b + 16 = 0$$

$$-2a + b + 4 = 0$$

$2a - b = 4 \quad —(1)$

$a = 5/2 \quad , \quad b = 1$

Substitute $(1,-1,2)$ in $ax^2 - byz = (a+2)x$

$a + 2b = a + 2$

$\boxed{b = 1}$

$a = 5/2$

Q:- $\nabla \times \nabla \times P$, where P is a vector is equal to

(a) $P \times \nabla \times P - \nabla^2 P$

(b) $\nabla^2 P + \nabla(\nabla \times P)$

(c) $\nabla^2 P + \nabla \times P$

(d) $\nabla(\nabla \cdot P) - \nabla^2 P$

- From vector triple product

$$A \times (B \times C) = (A \cdot C)\, B - (A \cdot B) \cdot C$$

$A \to \nabla$, $B \to \nabla$, $C \to P$

$$(\nabla \cdot P)\nabla - (\nabla \cdot \nabla) P$$

$$\nabla(\nabla P) - \nabla^2 P = (D)$$

Q:- Let ϕ be an arbitrary smooth real valued scalar function & V be an arbitrary smooth vector valued function in a 3-D space. Which one of following is an identity?

(a) $\mathrm{Curl}\,(\phi \vec{V}) = \nabla(\phi\, \mathrm{Div}\, \vec{V})$

(b) $\mathrm{Div}\, \vec{V} = 0$

(c) $\mathrm{Div}\,(\mathrm{curl}\, \vec{V}) = 0$ ✓

(d) $\mathrm{Div}\,(\phi \vec{V}) = \phi\, \mathrm{Div}\, \vec{V}$

Q:- The angle between two unit magnitude coplanar vectors P(0.866, 0.500, 0) and Q (0.259, 0.966, 0) ~~and~~ is ____.

(a) 0° (b) 30° (c) 45° ✓ (d) 60°

Sol:-

$$\bar{a} = 0.866\hat{i} + 0.500\hat{j} + 0\hat{k}$$

$$\bar{b} = 0.259\hat{i} + 0.966\hat{j} + 0\hat{k}$$

$$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$$

$$\bar{a}\cdot\bar{b} = (0.866)(0.259) + (0.5)(0.966) = 0.224 + 0.483 = 0.707$$

$$|\bar{a}| = \sqrt{(0.866)^2 + (0.5)^2} = \sqrt{0.7499 + 0.25} = 1$$

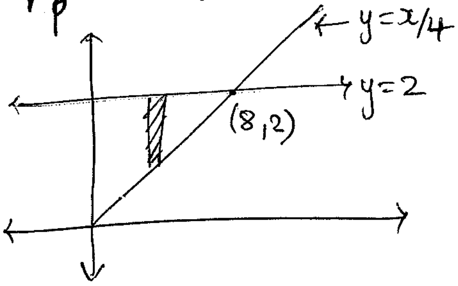
$$|\bar{b}| = \sqrt{(0.259)^2 + (0.966)^2} = \sqrt{0.0670 + 0.9331} = 1$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \frac{0.707}{1}$$

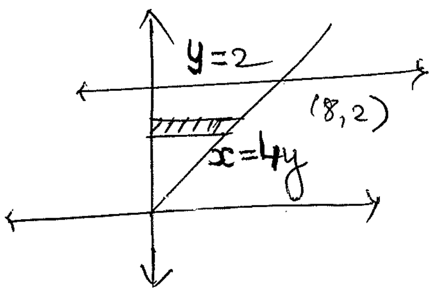
$$\boxed{\theta = 45^\circ}$$

Q:- Changing the order of integration in $I = \int_0^8 \int_{x/4}^{2} f(x,y)\,dy\,dx$ leads to $I = \int_r^s \int_p^q f(x,y)\,dx\,dy$. What is q?

Sol:-



Now q



$$\int_0^2 \int_0^{4y} f(x,y)\,dx\,dy$$

$$q = 4y$$

$\phi \Rightarrow$ scalar function

$$\text{grad}.\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$N = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$D.D. = \text{grad}\,\phi \cdot e$$

$$e = \frac{\vec{a}}{|\vec{a}|}$$

If f and g are two plane then angle betn two plane is $\cos\theta = \dfrac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|}$

② Divergence

Let $\vec{f} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any differentiable vector function then div.F is denoted by

div $\vec{F}$ OR $\nabla.F$ and is defined as

vector quantity

$$\boxed{\text{div}.F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

NOTE: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{div}.\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$= 3$$

$$(\text{grad } F) = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

$$\text{div}(\text{grad } F) = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = \nabla^2 f$$

Laplacian form

$= \nabla \cdot \nabla f$

Solenoid :- $\boxed{\text{div} . F = 0}$

Incompressible :- $\boxed{\text{div } F = 0}$

$\Rightarrow$ If ϕ = scalar, A = vector

$$\text{div}(\phi A) = (\text{grad } \phi)A + \phi(\text{div} . A)$$

Q:- Find $\nabla^2(1/r)$, $\nabla^2(\log r)$

~~grad F = $\frac{-1}{r^2}$~~

$$\nabla^2 F = \nabla \cdot (\nabla f)$$

$$= \nabla \cdot f' \frac{r}{|\vec{r}|}$$

$$= \nabla \cdot \left(\frac{-1}{r^2}\right) \cdot \frac{1/r}{|1/r|}$$

$$= \nabla \cdot \left[\frac{-1}{r^3}\right] \cdot \vec{r}$$

$\downarrow$ scalar $\quad$ $\downarrow$ vector

$$= \text{grad}\left(\frac{-1}{r^3}\right) \cdot \vec{r} + \frac{-1}{r^3}\,\text{div} . \vec{r}$$

$$= \frac{3}{r^4} \cdot \frac{\vec{r}}{r} \cdot \vec{r} + \frac{-1}{r^3}(3)$$

$$= \frac{3\cancel{r^2}}{r^{\cancel{5}3}} - \frac{3}{r^3}$$

$$\nabla^2 F = 0$$

2) $\nabla^2(\log r)$

$$\nabla^2 F = \nabla \cdot \nabla F$$

$$= \nabla \cdot \frac{1}{r}\,\frac{\vec{r}}{r}$$

$$= \nabla \; \frac{\vec{r} \rightarrow \text{vector}}{r^2 \rightarrow \text{scalar}}$$

$$= \text{grad}\left(\frac{1}{r^2}\right)\cdot \vec{r} + \left(\frac{1}{r^2}\right)\text{div}.\vec{r}$$

$$= \frac{-2}{r^3} \cdot \frac{\vec{r}}{r} \cdot \vec{r} + \frac{1}{r^2}\cdot 3$$

$$= \frac{-2\cancel{r^2}}{r^{\cancel{4}2}} + \frac{3}{r^2}$$

$$\nabla^2 F = \frac{1}{r^2}$$

$$\boxed{\underline{\text{Short-cut}} \quad \nabla^2 \cdot F(x) = f''(x) + \frac{2}{r} f'(x)}$$

(1) $\nabla^2\left(\frac{1}{r}\right) = \cancel{\frac{1}{r^2}}\ \frac{2}{r^3} + \frac{2}{r}\cdot\left(\frac{-1}{r^2}\right) \quad = 0$

(2) $\nabla^2(\log r) = \frac{-1}{r^2} + \frac{2}{r}\cdot\frac{1}{r} = \frac{1}{r^2}$

* Curl of vector function

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any differentiable vector function then curl of F is given as above

$\nabla \times F$.

☆ Irrotational: $\boxed{\nabla \times F = 0}$

☆

Q:- Find curl of $\vec{r}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \vec{r}=x & \vec{r}=y & \vec{r}=z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0$$

NOTE: $\boxed{\text{div}.(\text{curl} F) = 0}$

GATE-17

Q:- If vector function $\vec{F} = \hat{a}_x (3y - k_1 z) + \hat{a}_y (k_2 x - 2z) - \hat{a}_z (k_3 y + z)$ is irrotional, then the value of constant k_1, k_2, k_3 respectively are.

2m

curl F = 0 (for irrotional)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y-k_1 z & k_2 x - 2z & -(k_3 y + z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-k_3 y - z) - \frac{\partial}{\partial z}(k_2 x - 2z)\right] - \hat{j}\left[\frac{\partial(-k_3 y - z)}{\partial x} - \frac{\partial(3y - k_1 z)}{\partial z}\right]$$

$$+ \hat{k}\left[\frac{\partial(k_2 x - 2z)}{\partial x} - \frac{\partial}{\partial y}(3y - k_1 z)\right]$$

$$= \hat{i}\left[-k_3 + 2\right] - \hat{j}\left[0 + k_1\right] + \hat{k}\left[k_2 - 3\right]$$

$$k_1 = 0, \quad k_2 = 3, \quad k_3 = 2$$

ME-17

Q:- For vector $\vec{V} = 2yz\hat{i} + 3xz\hat{j} + 4xy\hat{k}$. The value of $\nabla \cdot (\nabla \times \vec{V}) = \_\_\_\_$.

2m

$= 0$

div. (curl) = 0

CE-17

Q:- The divergence of the vector field $V = x^2\hat{i} + 2y^3\hat{j} + z^4\hat{k}$ at $x = 1, y = 2$ & $z = 3$ is ____.

$\text{div}(\vec{V}) = 2x + 6y^2 + 4z^3$

$= 2 + 6(4) + 4(27) = 2 + 24 + 108 = \underline{134}$

Q:- The magnitude of directional derivative of function $f(x,y) = x^2+3y^2$ in direction normal to circle $x^2+y^2=2$ at point $(1,1)$

IN-15 Im

$$D.D. = \text{grad}\,\phi \cdot e$$

$$= (2x\hat{i} + 6y\hat{j}) \cdot \left[\frac{\vec{r}}{|\vec{r}|}\right]$$

$$= (2x\hat{i} + 6y\hat{j}) \cdot \left[\frac{2x\hat{i} + 2y\hat{j}}{\sqrt{8}}\right]$$

$$= (2\hat{i} + 6\hat{j}) \left[\frac{2x\hat{i} + 2y\hat{j}}{\sqrt{8}}\right]$$

$$= (2\hat{i} + 6\hat{j}) \left[\frac{2\hat{i} + 2\hat{j}}{\sqrt{8}}\right]$$

$e \xrightarrow{\text{on}} x^2+y^2=2$

↓

surface

for finding normal of

$$N = \frac{\text{grad}\, e}{|\text{grad}\, e|}$$

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4+4}}\bigg|_{(1,1)}$$

$$= \frac{2\hat{i} + 2\hat{j}}{\sqrt{8}}$$

$$D.D. = \frac{4+12}{\sqrt{8}} = \frac{16}{\sqrt{8}} = \frac{16}{2\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

$$\boxed{D.D. = 4\sqrt{2}}$$

Q:- For spherical surface $x^2+y^2+z^2=1$, the unit outward normal vector at the point $(1/\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$ is given by ___.

$$N = \frac{\text{grad}\, e}{|\text{grad}\, e|} = \cancel{\frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4+4+4}}}$$

$$= \cancel{\frac{\sqrt{2}\hat{i} + \sqrt{2}\hat{j}}{\sqrt{12}}} = \cancel{\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j}}$$

$$\text{grad } e = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$= \frac{2}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j}$$

$$\text{grad } e = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$$

$$N = \frac{\text{grad } e}{|\text{grad } e|} = \frac{\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}}{\sqrt{2+2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

EE

Q:- For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$. Magnitude of gradien

2m

at point (1, 3) is ____.

$$\text{grad } u = \frac{2x}{2}\hat{i} + \frac{2y}{3}\hat{j}$$

$$= \hat{i} + 2\hat{j}$$

$$|\text{grad } u| = \sqrt{1+4} = \sqrt{5}$$

Q:- Function $F = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$ then F

is (1) solenoid

(2) irrotional

✓(3) both (4) none

(1) solenoid ✓

div F = 0

0 + 0 + 0 = 0

(2) irrotional

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 0$$

Q:- The D.D. of function $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P(2,1,3)$ in the direction $a = \hat{i} - 2\hat{k}$ is ______.

D.D. = grad F · e

$$\text{grad } F = 2\cdot 2x\,\hat{i} + 3\cdot 2y\,\hat{j} + 2z\,\hat{k}$$

$$= 4(2)\hat{i} + 6(1)\hat{j} + 2(3)\hat{k}$$

$$= 8\hat{i} + 6\hat{j} + 6\hat{k}$$

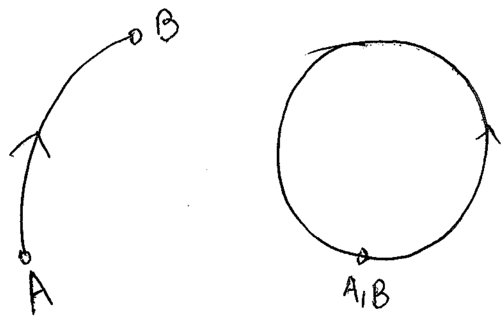
$$e = \frac{\vec{e}}{|\vec{e}|} = \frac{\hat{i} - 2\hat{k}}{\sqrt{1+4}}$$

$$D.D. = (8\hat{i} + 6\hat{j} + 6\hat{k}) \cdot \frac{\hat{i} - 2\hat{k}}{\sqrt{5}}$$

$$D.D. = \frac{8 - 12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

* Line Integral

- In a line integral we integrate the integrand along a curve c in space [or in the plane]



- A line integral of vector function $\vec{F}$ over a curve c is defined as

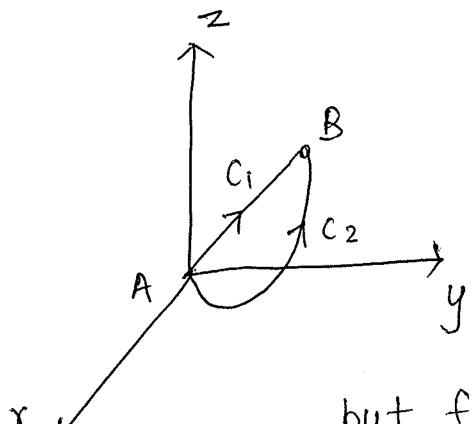
$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\boxed{\int_C F \cdot d\bar{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz}$$

$x, y, z = t$

$$\int_C F \cdot dx = \int_C F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}$$

- The value of line integral in general, will not only depend on $\vec{F}$ and end poin~~ets~~ a and b of the path but also along the path along which we integrate.



but for conservative field, it is importan
of path

Theorem: 1 [Independent of path]

A line integral with continuous F_1, F_2, F_3 in a domain D in space ~~in~~ is independent of path in D if and only if $F = [F_1, F_2, F_3]$ is the gradient of some function in D

$\boxed{F = \text{grad } f}$ ← potential

Theorem: 2

The line integral is independent of path in domain D if and only if it's value around very close path in D is zero.

Application of line integral:-

- The line integral $\int_C \vec{F}\cdot d\bar{r}$ represents work done by force F in the displacement of body along curve c.

IMP:

Q:- Evaluate $\int_C F\cdot d\vec{r}$ where $F = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$

(i) along the curve $x = 2t^2, y = t, z = t^3$ joining $(0,0,0)$ to $(2,1,1)$

(ii) The straight line joining $(0,0,0) \to (0,0,1) \to (0,1,1) \to (2,1,1)$

Sol:-

$$\int_C F\cdot dr = \int_{(0,0,0)}^{(2,1,1)} (2y+3)\,4t + xz\cdot 1 + (yz-x)(3t^2)\,dy$$

$$= \int_0^1 (2y+3)\cdot 4y + xz + (yz-x)\,3\left(\frac{x}{2}\right) dy$$

$$\int_0^1 \left(8y^2 + 12y + xz + \frac{3xyz}{2} - \frac{3x^2}{2}\right) dy$$

$$= \frac{8y^3}{3} + \frac{12y^2}{2} + xzy + \frac{3xy^2z}{4} - \frac{3x^2y}{2}$$

(i) $y = t$ [simplest range]

At $y = 0 \Rightarrow t = 0$

$y = 1 \Rightarrow t = 1$

$$\int_C F\,dx = \int_C F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}$$

$$dx = 4t\,dt \quad , \quad dy = dt \quad , \quad dz = 3t^2 dt$$

Now, $\int_0^1 ((2y+3)\,4t + xz \cdot 1 + (yz - x) \cdot 3t^2)\,dt$

$$\int_0^1 [(2t+3)(4t) + 2t^2 \cdot t^3 + (t t^3 - 2t^2) \cdot 3t^2]\,dt$$

$$\int_0^1 [8t^2 + 12t + 2t^5 + (t^4 - 2t^2)(3t^2)]\,dt$$

$$\left[\frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5}\right]_0^1$$

$$\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}$$

$$9 + \frac{15-42}{35}$$

$$9 + \frac{(-27)}{35}$$

$$\frac{315 - 27}{35} = \frac{288}{35}$$

(ii) $\overset{A}{(0,0,0)} \rightarrow \overset{B}{(0,0,1)} \longrightarrow \overset{C}{(0,1,1)} \longrightarrow \overset{D}{(2,1,1)}$

NOTE :- → Any line joinig two points can be evaluated using two ways when two co-ordinate are same and third is different ~~di~~ co-ordinates are the limits of line integral.

→ May any one are same & remaining are different even through we treat as different.

$$\int_C F\cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

A (0, 0, 0)
B (0, 0, 1)
C (0, 1, 1)
D (2, 1, 1)

(a) Along AB, $x = 0$, $y = 0$
$dx = 0$, $dy = 0$

$$\int_{AB} F\,dr = \int_0^1 (yz - x)\,dz = 0$$

(b) Along BC, $x = 0$, $z = 1$
$dx = 0$, $dz = 0$

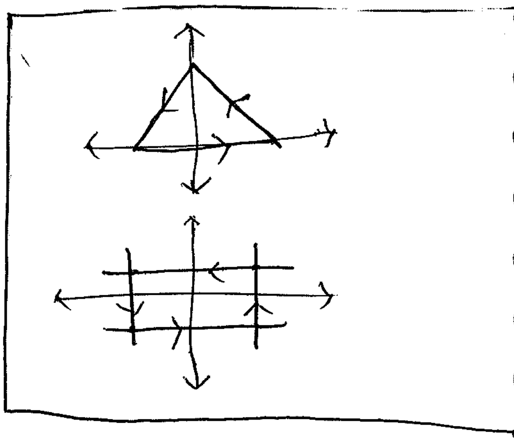
$$\int_{BC} F\,dr = \int_0^1 xy\cdot dy = 0$$

(c) Along CD, ~~x~~ $y = 1$, $z = 1$
$dy = 0$, $dz = 0$

$$\int_{CD} F\cdot dr = \int_0^2 (2y+3)\,dx$$

$$= \int_0^2 5\,dx$$

$$\int_{CD} F.dr = [5x]_0^2 = 10$$



$$\int_C F.dr = 0 + 0 + 10 = \underline{10}$$

Q. Evaluate line integral $\int_C F.d\bar{r}$ where
$F = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$
where c is the straight line joining (0, 0, 0) to (2, 1, 3).

$$\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$$

$$\overset{x_1\ y_1\ z_1}{(0,0,0)} \text{ to } \overset{x_2\ y_2\ z_2}{(2,1,3)}$$

straight line joining two points

* Symmetry form [parametric form]

$$A(x_1, y_1, z_1) \qquad B(x_2, y_2, z_2)$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t$$

$$\frac{x}{2} = y = \frac{z}{3} = t$$

$$x = 2t, \quad y = t, \quad z = 3t$$

$$\int_C F \cdot dr = \int F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}$$

$$= \int_{t=y=0}^{t=y=1} \left[3x^2 \cdot 2 + (2xz - y) \cdot 1 + z \cdot 3\right] dt$$

$$= \int_0^1 \left[6 \cdot 4t^2 + [2 \cdot 2t \cdot 3t - t] + 3 \cdot 3t\right] dt$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\, dt$$

$$\int_C F \cdot dr = \left[\frac{\overset{8}{\cancel{24}}t^3}{\cancel{3}} + \frac{\overset{4}{\cancel{12}}t^3}{\cancel{3}} + \frac{\overset{4}{\cancel{8}}t^2}{\cancel{2}}\right]_0^1 = \underline{16}$$

GATE
EC-17

Q:- Let $I = \int_C (2z\,dx + 2y\,dy + 2x\,dz)$ where x, y and z are real, and let c be straight line segment from $A(0,2,1)$ to $B(4,1,-1)$. The value of $I =$ ___ .

Sol:- $A(0,2,1)$ to $B(4,1,-1)$
$\quad x_1\ y_1\ z_1 \qquad\qquad x_2\ y_2\ z_2$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

$$\frac{x-0}{4-0} = \frac{y-2}{1-2} = \frac{z-1}{-1-1} = t$$

$$\frac{x}{4} = \frac{y-2}{-1} = \frac{z-1}{-2} = t$$

At $x=0 \Rightarrow t=0$
$x=4 \Rightarrow t=1$

$$x = 4t, \quad y = -t+2, \quad z = -2t+1$$

$$\int_C \vec{F}\cdot d\bar{r} = \int (2z)4 + 2y(-t+2) + 2x\cdot(-2t+1)\,dt$$

$$= \int_0^4 [2(-2t+1)4t + 2(-t+2)^2 + 2(4t)(-2t+1)]\,dt$$

$$= \int_0^4 [-16t^2 + 8t + 2[t^2-4t+4] - 16t^2 + 8t]$$

$$= \left[\frac{-16t^3}{3} + \frac{8t^2}{2} + \frac{2t^3}{3} - \frac{8t^2}{2} + 8t - \frac{16t^3}{3} + \frac{8t^2}{2}\right]_0^4$$

$$= [-10t^3 + 4t^2 + 8t]_0^4$$

$$= -10\cdot 64 + 4\cdot 16 + 8\cdot 4$$

$$= -640 + 64 + 32$$

$$\frac{dx}{dt} = 4 \;,\; \frac{dy}{dt} = -1 \;,\; \frac{dz}{dt} = -2$$

$$\int_0^1 2\cdot(-2t+1)4 + 2(-t+2)(-1) + 2(4t)(-2)\; dt$$

$$\int_0^1 \left(8(-2t+1) + 2(t-2) - 16t\right) dt$$

$$\int_0^1 (-16t + 8 + 2t - 4 - 16t)\, dt$$

$$\int_0^1 (-30t + 4)\, dt$$

$$\left[\frac{-30t^2}{2} + 4t\right]_0^1$$

$$\frac{-\cancel{30}^{15}}{\cancel{2}} \cdot 1 + 4 \cdot 1 = \underline{\underline{-11}}$$

Q:- Evaluate $F = 3xy\,\hat{i} - y^2\,\hat{j}$ where c is the curve in the x,y plane $y = 2x^2$ joining $(0,0)$ to $(1,2)$.

$$x = t \qquad y = 2t^2$$

$$\int_C F\cdot dr = \int_0^1 3xy\cdot\frac{dx}{dt} + (-y^2)\frac{dy}{dt}$$

$$= \int_0^1 3xy\cdot 1 - y^2(4t)$$

$$= \int_0^1 (3\cdot t\cdot 2t^2 - 4t^4\cdot 4t)\, dt$$

$$= \int_0^1 (6t^3 - 16t^5)\, dt = \left[\frac{\cancel{6}^{3}t^4}{\cancel{4}_{2}} - \frac{\cancel{16}^{8}t^6}{\cancel{6}_{3}}\right]_0^1 = \frac{3}{2} - \frac{8}{3}$$

$$= \frac{9-16}{6}$$

$$= -7/6$$

OR $\int_0^1 (3xy)dx + (-y^2)4x\,dx$

$= \int_0^1 3x \cdot 2x^2 \cdot dx - 4x^4 \cdot 4x\,dx$

$= \int_0^1 6x^3\,dx - 16x^5\,dx$

$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1 = \frac{-7}{6}$

Q:- Evaluate $\oint_c F \cdot d\bar{r}$ where $F = (2x - y + z)\hat{i} + (x + y - z)\hat{j} + (3x + 2y - 4z)\hat{k}$ where c is the circle in xy-plane at origin and radius 3 unit.

Sol:- XY-plane $z = 0$, $dz = 0$

$x^2 + y^2 = 9 = 3^2$

Limits: 0 to 2π

$x = r\cos\theta = 3\cos\theta$ | $y = r\sin\theta = 3\sin\theta$

$dx = -3\sin\theta\,d\theta$ | $dy = 3\cos\theta\,d\theta$

$\int_c F \cdot d\bar{r} = \int_0^{2\pi} \left[(2x - y + z)(-3\sin\theta) + (x + y - z) \cdot 3\cos\theta + 0\right] d\theta$

$= \int_0^{2\pi} \left(\left[2 \cdot 3\cos\theta - 3\sin\theta\right](-3\sin\theta) + (3\cos\theta + 3\sin\theta)(3\cos\theta)\right) d\theta$

$= \int_0^{2\pi} (-18\cos\theta\sin\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta)\,d\theta$

$$\int_0^{2\pi} (9 - 9\sin\theta\cos\theta)\,d\theta$$

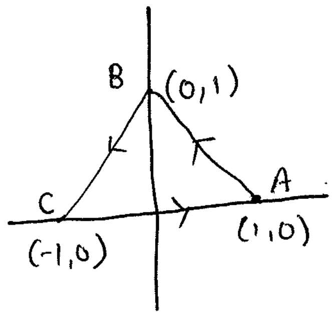
$$\int_0^{2\pi} \left(9 - 9\,\frac{\sin^2\theta}{2}\right) d\theta$$

$$\left[9\theta + \frac{9}{2}\cdot\frac{2\cos\theta}{2}\right]_0^{2\pi} = 18\pi + 9[\cos\pi - \cos 0]$$

$$\left[9\theta - \frac{9\sin^2\theta}{2}\right]_0^{2\pi} = 18\pi - 0 = \underline{18\pi}$$

Q:- Evaluate $\oint_c (y^2\,dx - x^2\,dy)$ where c is triangle formed by $(1,0)$, $(0,1)$ and $(-1,0)$.

Sol^n:- For AB



$$\frac{x-1}{0-1} = \frac{y-0}{1-0} = t$$

$$x-1 = -t, \quad y = t$$

$$x = -t+1$$

$$\int_{AB} y^2\cdot(-1) - x^2\cdot 1$$

Intercept form $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{x}{1} + \frac{y}{1} = 1$$

$$x + y = 1$$

$$1 + \frac{dy}{dx} = 0$$

$$dy = -dx$$

$$\int_{AB} y^2\,dx - x^2\cdot(-1)\,dx$$

$$\int_{AB} (1-x)^2\,dx + x^2\,dx$$

$$\int_1^0 (1 - 2x + 2x^2)\,dx$$

$$\left[x - x^2 + \frac{2x^3}{3}\right]_1^0$$

$$0 - [1 - 1 + 2/3] = -2/3$$

For BC,

$$\frac{x}{-1} + \frac{y}{1} = 1$$

$$-x + y = 1$$

$$-1 + \frac{dy}{dx} = 0$$

$$dy = dx$$

$$\int_{BC} y^2 dx - x^2 dx$$

$$\int_0^{-1} ((1+x)^2 - x^2) dx$$

$$\int_0^{-1} 1 + \cancel{x^2} + 2x - \cancel{x^2} = \left[x + \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^{-1}$$

$$= -1 + 1 - 0 = 0$$

For CA

$$y = 0$$

$$dy = 0$$

$$\int_{-1}^{1} y^2 dx - x^2 dy$$

$$= 0$$

$$\int_C y^2 dx - x^2 dy = \underline{\underline{-2/3}}$$

Q:- The line integral $\int \vec{V} \cdot d\vec{r}$ of vector $\vec{V}(r) = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$ from the origin to point $P(1,1,1)$

(a) 1 (c) -1

(b) 0 (d) Can't determine

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$$

$$z = x = y = t$$

$$\frac{dx}{dt} = 1, \ \frac{dy}{dt} = 1, \ \frac{dz}{dt} = 1$$

$$\int_C (2xyz)\,dt + (x^2 z)\,dt + (x^2 y)\,dt$$

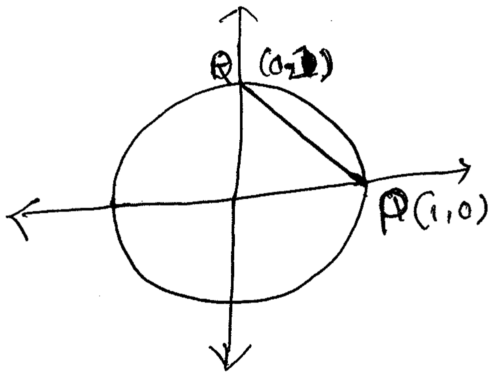
$$\int_0^1 (2t^3 + t^3 + t^3)\,dt$$

$$\int_0^1 4t^3 = \left[\frac{4t^4}{4}\right]_0^1 = \underline{\underline{1}}$$

2008-ECE

2m

Q:- Consider point P and Q in the xy-plane with $P=(1,0)$ and $Q=(0,1)$. The line integral $= 2\int_P^Q (x\,dx + y\,dy)$, along the semicircle with line segment PQ as its diameter is ____.



Sol:

$z = 0$

$dz = 0$

$$PQ = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$$

$$x^2 + y^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$2x\,dx + 2y\,dy = 0$$

$$2(x\,dx + y\,dy) = 0$$

$$2\int_P^Q x\,dx + y\,dy = \underline{\underline{0}}$$

Q:- The line integral of vector field $5xz\hat{i}+(3x^2+2y)\hat{j}+x^2z\hat{k}$
EE-16
2m along a path from (0,0,0) to (1,1,1) parameterized by (t, t^2, t) is ______.

$$\frac{x-0}{1-0}=t \,;\, \frac{y-0}{1-0}=t^2, \quad \frac{z-0}{1-0}=t$$

$$x=t, \quad y=t^2, \quad z=t.$$

$$\frac{dx}{dt}=1 \quad \frac{dy}{dt}=2t, \quad \frac{dz}{dt}=1$$

$$\int_C (5xz)\,dt+(3x^2+2y)(2t\,dt)+x^2z\cdot dt$$

$$\int_0^1 5t^2\,dt+(3t^2+2t^2)(2t\,dt)+t^2\cdot t\,dt$$

$$\int_0^1 (5t^2+10t^3+t^3)\,dt$$

$$\left[\frac{5t^3}{3}+\frac{11t^4}{4}\right]_0^1 = \frac{5}{3}+\frac{11}{4} = \frac{20+33}{12} = \frac{53}{12} = 4.41$$

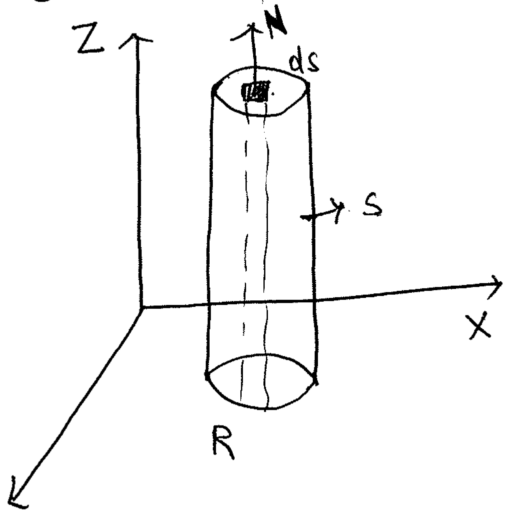
* Surface Integral

- Surface Integral, any Integral which is evaluated over a surface is called surface integral. Let F be any differentiable vector function defined over surface as then surface integral is defined as

$$\iint_S F\cdot N\,ds$$ where N is the outer unit normal vector to the given surface and is equal to $N=\frac{\nabla\phi}{|\nabla\phi|}$

NOTE:

To evaluate surface integral, projection is required in any two plane.



R is projection in xy plane

$$ds = \frac{dx\,dy}{|N\cdot\hat{k}|} = \frac{dy\,dz}{|N\cdot\hat{i}|} = \frac{dx\,dz}{|N\cdot\hat{j}|}$$

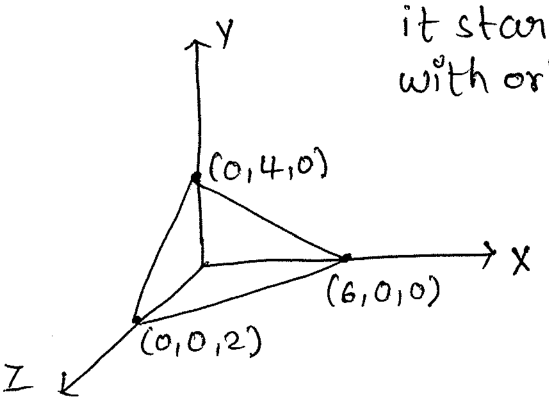
$$\iint_S F\cdot N\,ds = \iint_S F\cdot N \frac{dx\,dy}{|N\cdot\hat{k}|}$$

Q:- Find $\iint_S F\cdot N\,ds$ where $F = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and s is the part of plane $2x+3y+6z=12$ which is in 1st octant

↓ it starts with origin

Sol^n:-

$$N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}$$



$$N = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

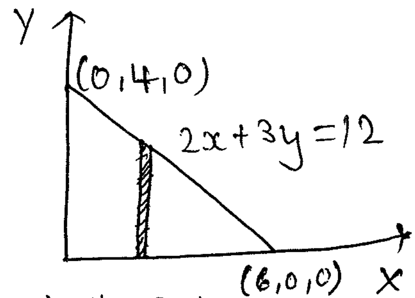
$$\iint_S F\cdot N\,ds = \iint_S (18z\hat{i} - 12\hat{j} + 3y\hat{k})\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)\cdot ds$$

$$= \iint_S \left(\frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}\right) ds$$

Now, $ds = \dfrac{dx\,dy}{|N\cdot\hat{k}|}$

$$|N\cdot\hat{k}| = \frac{6}{7}$$

Projection in xy plane



For x limits 6 to 0

For y limits take strips and eq^n z=0 so, $2x+3y=12$

$$= \iint_S \left(\frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}\right) \frac{dx\,dy}{6/7}$$

$$= \iint_S \frac{36z-36+18y}{\cancel{7}} \cdot \frac{dx\,dy}{6/\cancel{7}}$$

$$= \iint_S \frac{36z-36+18y}{6}\,dx\,dy$$

$$= \iint_S (6z-6+3y)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (-6+3y) \quad dy\,dx$$

$$= \int_0^6 \left[-6y + \frac{3}{2}y^2\right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \left(-6\left[\frac{12-2x}{3}\right] + \frac{3}{2}\left[\frac{12-2x}{3}\right]^2 - 0\right) dx$$

$$= \int_0^6 \left[-2(12-2x) + \frac{1}{6}(144-48x+4x^2)\right] dx$$

$$= \int_0^6 \left(-\cancel{24}+4x+\cancel{24}-8x+\frac{2}{3}x^2\right) dx$$

$$= \int_0^6 \left(\frac{2}{3}x^2 - 4x\right) dx$$

$$= \left[\frac{2}{3}\frac{x^3}{3} - \frac{\overset{2}{\cancel{4}}x^2}{2}\right]_0^6 = \frac{2}{\cancel{9}}\cdot\overset{2}{\cancel{6}}\cdot\overset{2}{\cancel{6}}\cdot 6 - 2\cdot 36$$

$$= 48 - 72 = |-24| = \underline{\underline{24}}$$

* Volume Integral

Any integral which is evaluated over a volume is called volume integral ($\iiint_V dV$)

* Vector Transformation

$$\boxed{\iiint (\text{div } F)\, dV}$$

① Gauss-divergence Theorem

$$\iint_S \longleftrightarrow \iiint_V$$

Let S be closed surface enclosed by volume V and let F be any differentiable vector function then

$$\boxed{\iint_S F\cdot N\, ds = \iiint_V \text{div}.\, F \cdot dV} \star$$

Q:- $\iint_S (ax\hat{i} + by\hat{j} + cz\hat{k})\cdot N\, ds$ where S is the sphere $x^2+y^2+z^2=1$. Evaluate surface integral.

$F = ax\hat{i} + by\hat{j} + cz\hat{k}$

$\text{div}.F = a\hat{i} + b\hat{j} + c\hat{k} = a+b+c$

From, Gauss-divergence Theorem,

$$\iint_S F\cdot N ds = \iiint_V \text{div}.F\, dV$$

$$\iiint_V \text{div} F \cdot dV$$

$$\iiint_V (a\hat{i} + b\hat{j} + c\hat{k})\, dx\, dy\, dz$$

$$(a+b+c)\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(a+b+c)$$

$r \to 1$

Q:- The $\iint_S \vec{r}\cdot N\,ds$ where s is the closed surface enclosed by volume V.

Sol:-

$$\text{div}\,\vec{F} = 1+1+1 = 3$$

$$\iiint_V 3\cdot dx\,dy\,dz = \underline{\underline{3V}}$$

Q:- $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ where S is the cube of unit length. Evaluate this.

$$\iint_S F\cdot N\cdot ds = \iiint_V \text{div}\,F\cdot dV \quad \left[\because \text{Gauss divergence theorem}\right]$$

$$\text{div}\,F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 1+1+1$$

$$F = 3$$

$$= \iiint_V 3\cdot dV$$

$$\iint_S F\cdot N\,ds = 3\,(1)^3 = \underline{\underline{3}}$$

Q:- Evaluate $\iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\,N\cdot ds$ where S is the cube bounded by $0 \le x, y, z \le 1$.

$$\text{div}\,F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 4z + (-2y) + y$$

$$\iiint_V (4z - 2y + y)\,dV$$

$$\int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^1 \left[\frac{4z^2}{2} - yz\right]_0^1 dy\,dx$$

$$= \int_0^1 \int_0^1 (2 - y)\, dy\, dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$2 - \frac{1}{2}$$

$$= \int_0^1 \frac{3}{2}\, dx$$

$$= \left[\frac{3}{2}x\right]_0^1 = \underline{\underline{\frac{3}{2}}}$$

Q:- $\iint_S F\cdot N\,ds$ where $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - 2x)\hat{j} + (z^2 - xy)\hat{k}$ where S is cuboid bounded by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

Sol^n:- $\text{div}\,\vec{F} = 2x + 2y + 2z$

$$\iiint_V \text{div}\,\vec{F} \cdot dV$$

$$= \int_0^a \int_0^b \int_0^c (2x + 2y + 2z)\, dz\, dy\, dx$$

$$= 2 \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dy\, dx$$

$$= 2 \int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2}\right) dy\, dx$$

$$= 2 \int_0^a \left[xcy + \frac{y^2}{2}c + \frac{c^2}{2}y\right]_0^b dx = 2\int_0^a \left(xcb + \frac{b^2}{2}c + \frac{c^2}{2}b\right) dx$$

$$= 2 \left[\frac{bcx^2}{2} + \frac{b^2cx}{2} + \frac{c^2bx}{2}\right]_0^a$$

$$= 2\left[\frac{bca^2}{2} + \frac{b^2ca}{2} + \frac{c^2ba}{2}\right]$$

$$= abc\,[a+b+c]$$

② GREEN'S THEOREM (IMP).

$\iint$ over a plane may be transformed into line integral over the boundary of regions and vice-versa

- Let R be a closed region in XY-plane bounded by a simple curve and enclosed (every closed non-intersecting curve is said to be simple and closed curve).

- Let P and Q are continuous function of x and y possess the first order partial derivative

closed surface is required. $\int_c \to \iint_s$

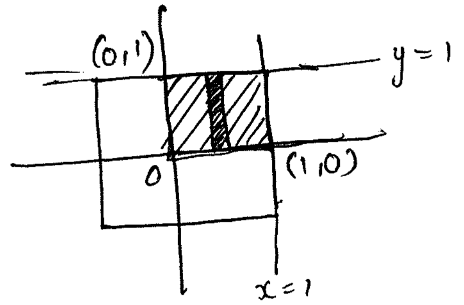
$$\boxed{\oint_c (P\,dx + Q\,dy) = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy}$$

CE-05

2m

Q:- The value of integral $\oint_c (xy\,dy - y^2\,dx)$ where c is the square cut from the first quadrant by the line $x=1$ & $y=1$ will be ____. [Using Green's theorem to change $\oint_c$ into $\iint$]

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$\frac{\partial Q}{\partial x} = y \quad , \quad \frac{\partial P}{\partial y} = -2y$$

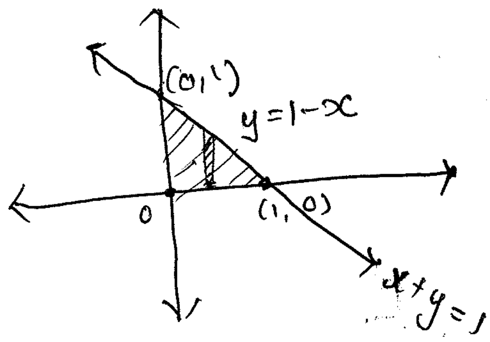
$$\int_0^1\int_0^1 (y + 2y)\,dy\,dx$$

$$\int_0^1 \left[\frac{y^2}{2} + y^2\right]_0^1 dx$$

$$\int_0^1 \frac{3}{2}\,dx = \left[\frac{3}{2}x\right]_0^1 = \frac{3}{2}$$

Q:- The value of $\oint_c (3x - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of region bounded by $x=0, y=0$ and $x+y=1$ is ____.

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$$



$$\frac{\partial Q}{\partial x} = -6y \quad , \quad \frac{\partial P}{\partial y} = -16y$$

$$= \int_0^1\int_0^{1-y} (-6y + 16y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{\cancel{10}\,5\,y^2}{\cancel{2}}\right]_0^{1-x} dx$$

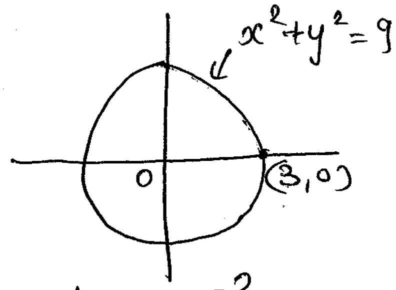
$$5(1-x)^2$$

$$= \int_0^1 5(1 + x^2 - 2x)\,dx \qquad \cancel{5x + xy^2 - 2yx}$$

$$= 5\left[x + \frac{x^3}{3} - x^2\right]_0^1 = 5\left[1 + \frac{1}{3} - 1\right] = 5/3 = 1.667$$

Q:- The value of $\oint_c (2x-y)\,dx + (x-y)\,dy$ where c is the circle $x^2+y^2=9$. Evaluate $\oint_c = ?$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$$



$$\frac{\partial Q}{\partial x} = 1 \quad , \quad \frac{\partial P}{\partial y} = -1$$

$ds = \pi R^2$

$$\iint_R (1-(-1))\,dy\,dx$$

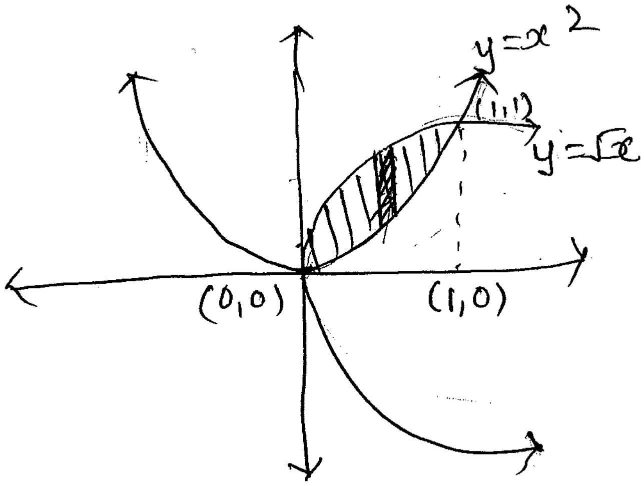
★ whenever fonction is constant then $\iint$ will give you directly surface area

$$\iint_R 2\,dy\,dx = 2\cdot\pi R^2 = 2\cdot\pi\cdot 3^2 = \underline{18\pi}$$

Q:- Evaluate $\oint_c (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where c is the region bounded by $y=\sqrt{x}$ & $y=x^2$

$$\oint_c P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} = -6y \quad , \quad \frac{\partial P}{\partial y} = -16y$$



$$\iint_R (10y)\,dy\,dx$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (10y)\,dy\,dx$$

$$\int_0^1 \left[\frac{10y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$\int_0^1 (5x - 5x^4)\,dx = \left[\frac{5x^2}{2} - \frac{5x^5}{5}\right]_0^1 = \frac{5}{2} - 1 = \frac{3}{2}$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (10y)\, dy\, dx$$

$$\int_0^1 [5y^2]_{x^2}^{\sqrt{x}}\, dx$$

$$\int_0^1 (5x - 5x^4)\, dx = \left[\frac{5x^2}{2} - x^5\right]_0^1 = \frac{5}{2} - 1 = \underline{\underline{\frac{3}{2}}}$$

(3) STOKES THEOREM

- Let S be an open surface bounded by closed and non-intersecting curve c and let F be any differentiable vector function then

$$\boxed{\int_c F \cdot d\bar{r} = \iint_S (\text{curl } F) \cdot N\, ds}$$

N = unit normal vector

Q:- Evalute by Stokes theorem $\oint_c F \cdot dr$ where $F = -y^3\hat{i} + x^3\hat{j}$ and s is the circular disc $x^2+y^2=1$, $z=0$.

$$\int_C F \cdot d\bar{r} = \iint_S (\text{curl } F) \cdot N\, ds$$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = \cancel{3x^2}(0-0)\hat{i} - \hat{j}(0-0) + \hat{k}(3x^2+3y^2)$$

$$\text{curl } F = (3x^2+3y^2)\hat{k} = 3\hat{k}$$

$$= \iint_S (\text{curl } F) \cdot N\, ds$$

$$= \iint 3\hat{k}\cdot\hat{k}\, ds = 3\iint ds = \underline{3\pi}$$

Q:- $\int_C \bar{r}\cdot dr$ where s is the region $x^2+y^2=1$ is ___.

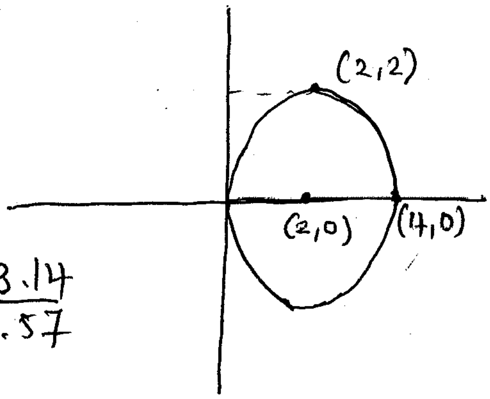
$$\text{curl } \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$ [curl of position vector is zero]

$$= \underline{0}$$

GATE-15

Q:- Consider ant crawling along the curve $(x-2)^2+y^2=4$ where x & y are in meter. The ant starts at the point (4,0) and move counter-clockwise with the speed of 1.57 m/s. The time taken by the ant to reach the point (2,2) in seconds is ___.

Sol^n:- $r = 2m$



$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

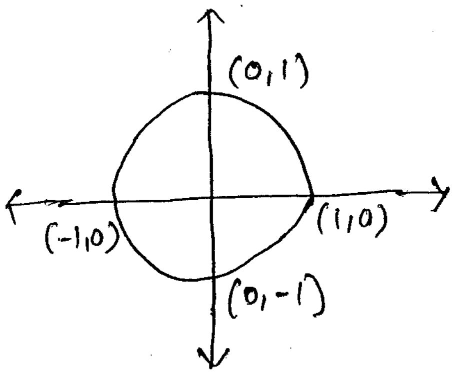
$$\text{Time} = \frac{2\pi R}{4\cdot 1.57} = \frac{2\pi\cdot\cancel{2}}{\cancel{4}\cdot 1.57} = \frac{3.14}{1.57}$$

$$\underline{\text{Time} = 2s}$$

EE-14

Q:- The line integral of function $F = yz\hat{i}$, in the counter clock-wise direction along the circle $x^2+y^2=1$ at $z=1$ is ___.

Sol:- $\int_C \bar{F}\cdot d\bar{r} = \iint_S (\text{curl}F)\cdot N\, ds$



$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 0 & 0 \end{vmatrix} \quad -\hat{j}(0-y)$$

$$= 0 + \cancel{0} + \hat{k}(0-z)$$

$$= -z\hat{k}+y\hat{j} = -\hat{k}z + y\hat{j}$$

$$\iint_S (\text{curl }F)\cdot N\, ds = \iint_S \begin{pmatrix}-\hat{k}z \\ +y\hat{j}\end{pmatrix}\hat{k}\, ds = \iint_S -z\, ds = -\pi R^2$$

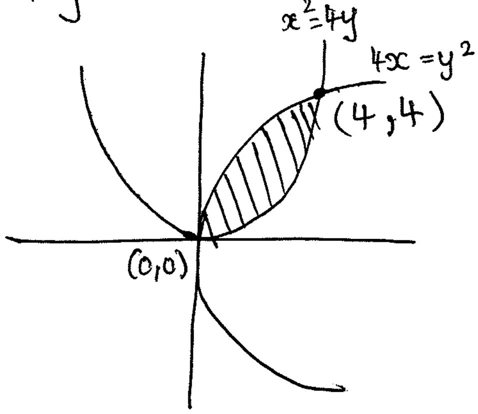
$$= -\pi\cdot 1$$

$$= -\pi$$

*Application of Integration

① Area between the curves

Q:- The area enclosed by curve $y^2 = 4x$ and $x^2 = 4y$



$y^2 = 4(2\sqrt{y})$

$y^2 - 8\sqrt{y} = 0$

$y^4 = 64y$

$y = 0 \quad , \quad y^3 = 64$

$y = 4$

$$= \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy \; dx$$

$$= \int_0^4 [y]_{x/2}^{2\sqrt{x}} dx$$

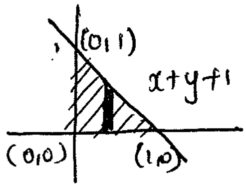
$$= \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4} \right] dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{x^3}{4 \cdot 3} \right]_0^4$$

$$= \frac{4}{3}(4)^{3/2} - \frac{4 \cdot \cancel{4} \cdot 4}{\cancel{4} \cdot 3}$$

$$= \frac{4}{3}\, 4 \cdot 2 - \frac{16}{3}$$

$$= \frac{16 \cdot 2 - 16}{3} = \frac{32 - 16}{3} = \frac{16}{3}$$



$$\int_0^1 \int_0^{1-x} dy \, dx$$

OR

$$\int_0^1 (\text{upper curve}) - \text{lower curve}$$

$$\int_0^1 (1-x) - 0 \; dx$$

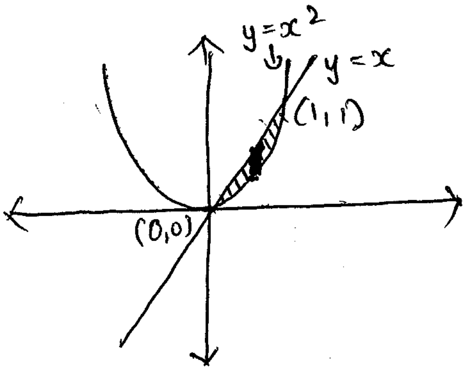
NOTE: $y^2=4ax$ and $x^2=4ay$ then area betⁿ two curves (parabolas).

$$\text{AREA} = \frac{16a^2}{3}$$

(i) $y^2=8x$ & $x^2=8y$ $\Rightarrow$ $\frac{16\cdot 4}{3} = \frac{64}{3}$

(ii) $y^2=x$ & $x^2=y$ $\Rightarrow$ $\frac{16\cdot\frac{1}{16}}{3} = \frac{1}{3}$

Q:- The area enclosed between the straight lines $y=x$ and parabola $y=x^2$ in XY-plane is ____.



$y=x^2$

$x=x^2$

$x^2-x=0$

$x(x-1)=0$

$x=0,\ x=1$

$y=0,\ y=1$

$$\text{AREA} = \int_0^1\int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx$$

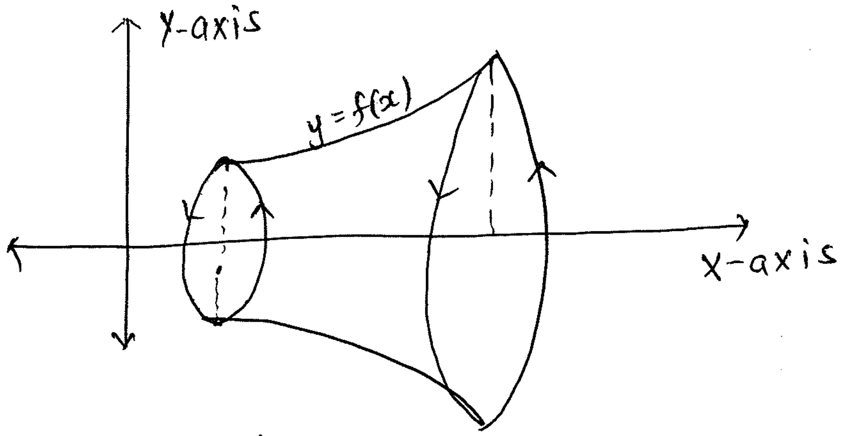
$$= \int_0^1 (x-x^2)\,dx$$

$$= \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{3-2}{6} = \underline{\underline{\frac{1}{6}}}$$

② Volume of revolution

- Revolution about X-axis

The volume of solid generated by revolution about X-axis, of the area bounded by curve $y=f(x)$ and co-ordinate $x=a$, $x=b$ is $\int_a^b \pi y^2\,dx$



- Revolution of Y-axis

$$\underset{\text{volume}}{V} = \int_a^b \pi x^2 \, dy$$

Q:- The parabolic arc $y = \sqrt{x}$, x varies from $1 \le x \le 2$ is revoled around X-axis. The volume of solid of revolution is ____.

$$V = \int_a^b \pi y^2 \, dx$$

$$V = \int_1^2 \pi \cdot x \, dx$$

$$V = \pi \left[\frac{x^2}{2}\right]_1^2 = \left(\frac{4}{2} - \frac{1}{2}\right)\pi = \underline{\underline{\frac{3\pi}{2}}}$$

③ Length of curve

- The length of arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is

$$\boxed{L = \int_a^b \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\right] dx}$$

- The length of arc of the curve $x = f(y)$ between the point $y = a$ and $y = b$ is

$$L = \int_a^b \left(\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,\right) dy$$

- The length of the arc of the curve $x = f(t)$ and y is also the function of t ($y = f(t)$) between the point $t = a$ and $t = b$ is.

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Q:- The length of the curve $y = \frac{2}{3} x^{3/2}$ between $x = 0$ and $x = 1$ is ______.

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \left(\frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} x^{1/2}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + x}\, dx \qquad \cancel{\text{LIATE}}$$

$$\cancel{= \left[\frac{1}{2\sqrt{1+x}}\right]_0^1 = \frac{1}{2\sqrt{2}} - \frac{1}{2}} = \left[\frac{2(1+x)^{3/2}}{3}\right]_0^1$$

$$\cancel{= \frac{1}{2\sqrt{1+x}}}$$

$$= \frac{2}{3} \cdot 2^{3/2} - \frac{2}{3}$$

$$= \frac{2}{3}\left(2\sqrt{2} - 1\right)$$