Chapter 16

THE GENERAL CONIC

370. In the present chapter we shall consider properties of conic sections which are given by the general equation of the second degree, viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots (1).$$

For brevity, the left-hand side of this equation is often called $\phi(x, y)$, so that the general equation to a conic is

$$\phi(x, y) = 0.$$

Similarly, $\phi(x', y')$ denotes the value of the left-hand side of (1) when x' and y' are substituted for x and y.

The equation (1) is often also written in the form $S = 0$.

371. On dividing by c, the equation (1) contains five independent constants $\frac{a}{c}$, $\frac{h}{c}$, $\frac{b}{c}$, $\frac{g}{c}$, and $\frac{f}{c}$.

To determine these five constants, we shall therefore require five conditions. Conversely, if five independent conditions be given, the constants can be determined. Only one conic, or, at any rate, only a finite number of conics, can be drawn to satisfy five independent conditions.

372. *To find the equation to the tangent at any point (x', y') of the conic section*

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots (1).$$

Let (x'', y'') be any other point on the conic.

The equation to the straight line joining this point to (x', y') is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \ldots\ldots\ldots\ldots\ldots (2).$$

Since both (x', y') and (x'', y'') lie on (1), we have

$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0 \ldots\ldots (3),$$

and $$ax''^2 + 2hx''y'' + by''^2 + 2gx'' + 2fy'' + c = 0 \ldots\ldots (4).$$

Hence, by subtraction, we have

$$a(x'^2 - x''^2) + 2h(x'y' - x''y'') + b(y'^2 - y''^2)$$
$$+ 2g(x' - x'') + 2f(y' - y'') = 0 \ldots\ldots\ldots\ldots (5).$$

But

$$2(x'y' - x''y'') = (x' + x'')(y' - y'') + (x' - x'')(y' + y''),$$

so that (5) can be written in the form

$$(x' - x'')[a(x' + x'') + h(y' + y'') + 2g]$$
$$+ (y' - y'')[h(x' + x'') + b(y' + y'') + 2f] = 0,$$

i.e. $$\frac{y'' - y'}{x'' - x'} = -\frac{a(x' + x'') + h(y' + y'') + 2g}{h(x' + x'') + b(y' + y'') + 2f}.$$

The equation to any secant is therefore

$$y - y' = -\frac{a(x' + x'') + h(y' + y'') + 2g}{h(x' + x'') + b(y' + y'') + 2f}(x - x') \ldots (6).$$

To obtain the equation to the tangent at (x', y'), we put $x'' = x'$ and $y'' = y'$ in this equation, and it becomes

$$y - y' = -\frac{ax' + hy' + g}{hx' + by' + f}(x - x'),$$

i.e. $$(ax' + hy' + g)x + (hx' + by' + f)y$$
$$= ax'^2 + 2hx'y' + by'^2 + gx' + fy'$$
$$= -gx' - fy' - c, \text{ by equation (3).}$$

The required equation is therefore

$$\mathbf{axx' + h(xy' + x'y) + byy' + g(x + x') + f(y + y')}$$
$$\mathbf{+ c = 0} \ldots\ldots (7).$$

Cor. 1. The equation (7) may be written down, from the general equation of the second degree, by substituting xx' for x^2, yy' for y^2, $xy' + x'y$ for $2xy$, $x + x'$ for $2x$, and $y + y'$ for $2y$. (Cf. Art. 152.)

Cor. 2. If the conic pass through the origin we have $c=0$, and then the tangent at the origin (where $x'=0$ and $y'=0$) is
$$gx+fy=0,$$
i.e. the equation to the tangent at the origin is obtained by equating to zero the terms of the lowest degree in the equation to the conic.

373. The equation of the previous article may also be obtained as follows; If (x', y') and (x'', y'') be two points on the conic section, the equation to the line joining them is
$$a(x-x')(x-x'')+h[(x-x')(y-y'')+(x-x'')(y-y')]+b(y-y')(y-y'')$$
$$=ax^2+2hxy+by^2+2gx+2fy+c \ldots\ldots\ldots(1).$$

For the terms of the second degree on the two sides of (1) cancel, and the equation reduces to one of the first degree, thus representing a straight line.

Also, since (x', y') lies on the curve, the equation is satisfied by putting $x=x'$ and $y=y'$.

Hence (x', y') is a point lying on (1).

So (x'', y'') lies on (1).

It therefore is the straight line joining them.

Putting $x''=x'$ and $y''=y'$ we have, as the equation to the tangent at (x', y'),
$$a(x-x')^2+2h(x-x')(y-y')+b(y-y')^2$$
$$=ax^2+2hxy+by^2+2gx+2fy+c,$$
i.e.
$$2axx'+2h(x'y+xy')+2byy'+2gx+2fy+c$$
$$=ax'^2+2hx'y'+by'^2$$
$$=-2gx'-2fy'-c, \text{ since } (x', y') \text{ lies on the conic.}$$

Hence the equation (7) of the last article.

374. *To find the condition that any straight line*
$$lx+my+n=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
may touch the conic
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots\ldots\ldots\ldots (2).$$

Substituting for y in (2) from (1), we have for the equation giving the abscissæ of the points of intersection of (1) and (2),
$$x^2(am^2-2hlm+bl^2)-2x(hmn-bln-gm^2+flm)$$
$$+bn^2-2fmn+cm^2=0 \ldots\ldots\ldots(3).$$

If (1) be a tangent, the values of x given by (3) must be equal.

The condition for this is, (Art. 1,)
$$(hmn-bln-gm^2+flm)^2=(am^2-2hlm+bl^2)(bn^2-2fmn+cm^2).$$

On simplifying, we have, after division by m^2,

$$l^2(bc-f^2)+m^2(ca-g^2)+n^2(ab-h^2)+2mn(gh-af)+2nl(hf-bg)$$
$$+2lm(fg-ch)=0.$$

Ex. *Find the equations to the tangents to the conic*

$$x^2+4xy+3y^2-5x-6y+3=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which are parallel to the straight line $x+4y=0$.

The equation to any such tangent is

$$x+4y+c=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where c is to be determined.

This straight line meets (1) in points given by

$$3x^2-2x(5c+28)+3c^2+24c+48=0.$$

The roots of this equation are equal, *i.e.* the line (2) is a tangent, if $\{2(5c+28)\}^2=4\,.\,3\,.\,(3c^2+24c+48)$, *i.e.* if $c=-5$ or -8.

The required tangents are therefore

$$x+4y-5=0, \text{ and } x+4y-8=0.$$

375. As in Arts. 214 and 274 it may be proved that the polar of (x', y') with respect to $\phi(x, y)=0$ is

$$(ax'+hy'+g)x+(hx'+by'+f)y+gx'+fy'+c=0.$$

The form of the equation to a polar is therefore the same as that of a tangent.

Just as in Art. 217 it may now be shewn that, if the polar of P passes through T, the polar of T passes through P.

The chord of the conic which is bisected at (x', y'), being parallel to the polar of (x', y') [Arts. 221 and 280], has as equation

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')=0.$$

376. *To find the equation to the diameter bisecting all chords parallel to the straight line $y=mx$.* (See fig. Art. 279.)

Any such chord is $\quad y=mx+K \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$

This meets the conic section

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

in points whose abscissæ are given by

$$ax^2+2hx(mx+K)+b(mx+K)^2+2gx+2f(mx+K)+c=0,$$

i.e. by $x^2(a+2hm+bm^2)+2x(hK+bmK+g+fm)$

$$+bK^2+2fK+c=0.$$

If x_1 and x_2 be the roots of this equation, we therefore have

$$x_1 + x_2 = -2\frac{(h+bm)K + g + fm}{a + 2hm + bm^2}.$$

Let (X, Y) be the middle point of the required chord, so that

$$X = \frac{x_1 + x_2}{2} = -\frac{(h+bm)K + g + fm}{a + 2hm + bm^2} \quad \text{.........(2).}$$

Also, since (X, Y) lies on (1) we have

$$Y = mX + K \quad \text{.................. .. (3).}$$

If between (2) and (3) we eliminate K we have a relation between X and Y.

This relation is

$$-(a + 2hm + bm^2)X = (h + bm)(Y - mX) + g + fm,$$

i.e.
$$X(a + hm) + Y(h + bm) + g + fm = 0.$$

The locus of the required middle point is therefore the straight line whose equation is

$$x(a + hm) + y(h + bm) + g + fm = 0.$$

If this be parallel to the straight line $y = m'x$, we have

$$m' = -\frac{a + hm}{h + bm} \quad \text{....................(4),}$$

i.e.
$$\mathbf{a + h(m + m') + bmm' = 0} \quad \text{............... (5).}$$

This is therefore the condition that the two straight lines $y = mx$ and $y = m'x$ may be parallel to conjugate diameters of the conic given by the general equation.

377. *To find the condition that the pair of straight lines, whose equation is*

$$Ax^2 + 2Hxy + By^2 = 0 \quad \text{........................ (1),}$$

may be parallel to conjugate diameters of the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{............... (2).}$$

Let the equations of the straight lines represented by (1) be $y = mx$ and $y = m'x$, so that (1) is equivalent to

$$B(y - mx)(y - m'x) = 0,$$

and hence
$$m + m' = -\frac{2H}{B}, \text{ and } mm' = \frac{A}{B}.$$

By the condition of the last article it therefore follows that the lines (1) are parallel to conjugate diameters if

$$a+h\left(-\frac{2H}{B}\right)+b\frac{A}{B}=0,$$

i.e. if

$$Ab-2Hh+Ba=0.$$

378. *To prove that two concentric conic sections always have a pair, and only one pair, of common conjugate diameters and to find their equation.*

Let the two concentric conic sections be

$$ax^2+2hxy+by^2=1 \qquad (1),$$

and

$$a'x^2+2h'xy+b'y^2=1 \qquad (2).$$

The straight lines

$$Ax^2+2Hxy+By^2=0 \qquad (3),$$

are conjugate diameters of both (1) and (2) if

$$Ab-2Hh+Ba=0,$$

and

$$Ab'-2Hh'+Ba'=0.$$

Solving these two equations we have

$$\frac{A}{ha'-h'a}=\frac{-2H}{ab'-a'b}=\frac{B}{bh'-b'h}.$$

Substituting these values in (3), we see that the straight lines

$$x^2(ha'-h'a)-xy(ab'-a'b)+y^2(bh'-b'h)=0 \qquad (4)$$

are always conjugate diameters of both (1) and (2).

Hence there is always a pair of conjugate diameters, real, coincident, or imaginary, which are common to any two concentric conic sections.

EXERCISES XLII

1. How many other conditions can a conic section satisfy when we are given (1) its centre, (2) its focus, (3) its eccentricity, (4) its axes, (5) a tangent, (6) a tangent and its point of contact, (7) the position of one of its asymptotes?

2. Find the condition that the straight line $lx+my=1$ may touch the parabola $(ax-by)^2-2(a^2+b^2)(ax+by)+(a^2+b^2)^2=0$, and shew that if this straight line meet the axes in P and Q, then PQ will, when it is a tangent, subtend a right angle at the point (a, b).

3. Two parabolas have a common focus; prove that the perpendicular from it upon the common tangent passes through the intersection of the directrices.

4. Shew that the conic $\frac{x^2}{a^2}+\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$ is inscribed in the rectangle, the equations to whose sides are $x^2=a^2$ and $y^2=b^2$, and that the quadrilateral formed by joining the points of contact is of constant perimeter $4\sqrt{a^2+b^2}$, whatever be the value of α.

5. A variable tangent to a conic meets two fixed tangents in two points, P and Q; prove that the locus of the middle point of PQ is a conic which becomes a straight line when the given conic is a parabola.

6. Prove that the chord of contact of tangents, drawn from an external point to the conic $ax^2+2hxy+by^2=1$, subtends a right angle at the centre if the point lie on the conic

$$x^2(a^2+h^2)+2h(a+b)xy+y^2(h^2+b^2)=a+b.$$

7. Given the focus and directrix of a conic, prove that the polar of a given point with respect to it passes through another fixed point.

8. Prove that the locus of the centres of conics which touch the axes at distances a and b from the origin is the straight line $ay=bx$.

9. Prove that the locus of the poles of tangents to the conic $ax^2+2hxy+by^2=1$ with respect to the conic $a'x^2+2h'xy+b'y^2=1$ is the conic

$$a(h'x+b'y)^2-2h(a'x+h'y)(h'x+b'y)+b(a'x+h'y)^2=ab-h^2.$$

10. Find the equations to the straight lines which are conjugate to the coordinate axes with respect to the conic $Ax^2+2Hxy+By^2=1$.

Find the condition that they may coincide, and interpret the result.

11. Find the equation to the common conjugate diameters of the conics (1) $x^2+4xy+6y^2=1$ and $2x^2+6xy+9y^2=1$,

and (2) $2x^2-5xy+3y^2=1$ and $2x^2+3xy-9y^2=1$.

12. Prove that the points of intersection of the conics

$$ax^2+2hxy+by^2=1 \text{ and } a'x^2+2h'xy+b'y^2=1$$

are at the ends of conjugate diameters of the first conic, if

$$ab'+a'b-2hh'=2(ab-h^2).$$

13. Prove that the equation to the equi-conjugate diameters of the conic $ax^2+2hxy+by^2=1$ is $\frac{ax^2+2hxy+by^2}{ab-h^2}=\frac{2(x^2+y^2)}{a+b}$.

ANSWERS

1. (1) 3; (2) 3; (3) 4; (4) 2; (5) 4; (6) 3; (7) 3.

10. $Ax+Hy=0$ and $Hx+By=0$; $H^2=AB$, so that the conic is a pair of parallel straight lines.

11. $x(x+3y)=0$; $(2x-3y)^2=0$.

SOLUTIONS/HINTS

1. (1) The equation of any conic, whose centre is the origin, is $ax^2 + 2hxy + by^2 = 1$.

Therefore three other conditions are required to determine a, h, b.

(2) The equation of any conic, one of whose foci is the origin, is $x^2 + y^2 = e^2 (x \cos \alpha + y \sin \alpha - p)^2$.

Therefore three other conditions are required to determine e, α, p.

(3) The equation of any conic, one of whose foci is (h, k), is

$$(x-h)^2 + (y-k)^2 = e^2 (x \cos \alpha + y \sin \alpha - p)^2.$$

Four other conditions are required to determine h, k, α, p.

(4) The equation of any conic whose axes are the axes of coordinates is $ax^2 + by^2 = 1$.

Therefore two other conditions are required to determine a, b.

(5) By Art. 374, a conic can be determined to touch five given lines; therefore four other conditions are required.

(6) A tangent and its point of contact are equivalent to *two* consecutive points of the curve; therefore three other conditions are required.

(7) By Art. 326 (1), if $Ax + By + C = 0$ is given, three other conditions are required to determine $A_1x + B_1y + C_1 = 0$ and λ^2.

Aliter. To give the position of one asymptote is the same as to give *two* points on the curve at infinity. Hence three other conditions are necessary, since a conic can be drawn through any five points.

2. The lines joining the origin to the common points of the parabola and the line are

$$(ax - by)^2 - 2(a^2 + b^2)(ax + by)(lx + my) + (a^2 + b^2)^2 (lx + my)^2 = 0. \quad \text{[Art. 122.]}$$

If $lx + my = 1$ is a tangent, these must be coincident;

$$\therefore \ \{a - l(a^2+b^2)\}^2 \{b - m(a^2+b^2)\}^2$$
$$= [\{a - l(a^2+b^2)\}\{b - m(a^2+b^2)\} - 2ab]^2.$$
$$\therefore \ \{a - l(a^2+b^2)\}\{b - m(a^2+b^2)\} = ab.$$
$$\therefore \ lm(a^2+b^2) = am + lb.$$

This is the same as the condition that PQ will subtend a right angle at (a, b), viz.

$$\frac{b}{a - \frac{1}{l}} \cdot \frac{b - \frac{1}{m}}{a} + 1 = 0.$$

3. Taking the common focus for origin, let

$$x^2 + y^2 = (x\cos\alpha + y\sin\alpha - p_1)^2,$$
$$x^2 + y^2 = (x\cos\beta + y\sin\beta - p_2)^2,$$

be the equations of the parabolas.

$lx + my + 1 = 0$ is a common tangent if

$(l^2 + m^2)p_1 + 2m\sin\alpha + 2l\cos\alpha = 0$, ... (i)

$(l^2 + m^2)p_2 + 2m\sin\beta + 2l\cos\beta = 0$. ... (ii) [Art. 374.]

Multiply (ii) by p_1, (i) by p_2, and subtract. Therefore

$$p_2(m\sin\alpha + l\cos\alpha) - p_1(m\sin\beta + l\cos\beta) = 0. \ \ldots \text{(iii)}$$

The equation of the line through the origin, perpendicular to $lx + my + 1 = 0$,

is $$\frac{x}{l} = \frac{y}{m}.$$

Hence from (iii) it is

$$p_2(x\cos\alpha + y\sin\alpha - p_1) - p_1(x\cos\beta + y\sin\beta - p_2) = 0$$

which is the equation of a line going through the intersection of the directrices.

4. The four lines $x = \pm a$, $y = \pm b$ each cut the curve in coincident points, whose coordinates are

$(a, -b\cos\alpha)$, $(-a\cos\alpha, b)$, $(-a, b\cos\alpha)$, $(a\cos\alpha, -b)$.

The perimeter

$$=2\{a^2(1+\cos\alpha)^2+b^2(1+\cos\alpha)^2\}^{\frac{1}{2}}$$
$$+2\{a^2(1-\cos\alpha)^2+b^2(1-\cos\alpha)^2\}^{\frac{1}{2}}=4\sqrt{a^2+b^2}.$$

5. Taking the general equation of Art. 372, if $x=0$, $y=0$ are tangents, we have $g^2=ac$, $f^2=bc$, and the condition of Art. 374 becomes, if $n=1$,

$$(ab-h^2)+2m(gh-af)+2l(hf-bg)+2lm(fg-ch)=0;$$

but if (x, y) be the coordinates of the middle point of PQ,

$$2l=\frac{1}{x},\quad 2m=\frac{1}{y}.$$

Hence the equation of the required locus is the conic

$$(ab-h^2)xy+(gh-af)x+(hf-bg)y+\tfrac{1}{2}(fg-ch)=0,$$

which becomes a straight line if $ab-h^2=0$.

6. The lines joining the origin to the common points of

$$ax^2+2hxy+by^2=1 \text{ and } x(ax'+hy')+y(hx'+by')=1$$

are $\quad ax^2+2hxy+by^2=\{x(ax'+hy')+y(hx'+by')\}^2.$

[Art. 122.]

They are at right angles if

$$a+b=x'^2(a^2+h^2)+2h(a+b)x'y'+y'^2(h^2+b^2).$$

Suppressing the accents, we have the required locus.

7. Take the focus for origin and let $x=a$ be the equation of the corresponding directrix. Then the equation of the conic will be $x^2+y^2=e^2(x-a)^2$.

The polar of the point (x', y') is

$$xx'+yy'-e^2\{xx'-a(x+x')+a^2\}=0,\quad \text{[Art. 375]}$$

which passes through the intersection of the lines

$$xx'+yy'=0, \text{ and } xx'-a(x+x')+a^2=0.$$

8. The equation of the conic is

$$b^2x^2+2\lambda^2xy+a^2y^2-2b^2ax-2a^2by+a^2b^2=0.$$

For when $y=0$, we have $(x-a)^2=0$, and when $x=0$, we have $(y-b)^2=0$.

The centre is given by

$$b^2x + \lambda^2 y = b^2 a, \quad \lambda^2 x + a^2 y = a^2 b. \quad \text{[Art. 352.]}$$

Eliminating λ, $\quad b^2x^2 - a^2y^2 = ab\,(bx - ay)$.

Therefore $ay = bx$, since $\frac{x}{a} + \frac{y}{b} = 1$ is clearly not part of the locus, or rather it is the locus for the particular case when $\lambda^2 = ab$, *i.e.* when the conic is two coincident straight lines coinciding with $\frac{x}{a} + \frac{y}{b} = 1$.

9. The polar of (x', y') with regard to the second conic, viz. $\quad x\,(a'x' + h'y') + y\,(h'x' + b'y') = 1$,
will touch the first conic if

$$a\,(h'x' + b'y')^2 - 2h\,(h'x' + b'y')\,(a'x' + h'y') + b\,(a'x' + h'y')^2 = ab - h^2. \quad \text{[Art. 374.]}$$

Suppressing the accents, we have the required equation of the locus.

10. (1) Let $y - mx = 0$ be conjugate to $y = 0$. Then since $y^2 - mxy = 0$, are conjugate diameters of

$$Ax^2 + 2Hxy + By^2 = 1, \quad \therefore\ A + mH = 0. \quad \text{[Art. 377.]}$$

Therefore $Hy + Ax = 0$ is conjugate to $y = 0$.

Similarly $Hx + By = 0$ is conjugate to $x = 0$.

(2) If they coincide, $\frac{A}{H} = \frac{H}{B}$, or $H^2 = AB$, so that the conic becomes a pair of parallel straight lines.

11. By Equation (4) of Art. 378, the required equations are

$$(1) \quad x\,(x + 3y) = 0 \ \text{ and } \ (2) \quad (2x - 3y)^2 = 0.$$

12. The lines joining the origin to the common points of the conics are clearly

$$(a - a')\,x^2 + 2\,(h - h')\,xy + (b - b')\,y^2 = 0.$$

These are conjugate diameters of the first conic if

$$a\,(b - b') - 2h\,(h - h') + (a - a')\,b = 0,$$

i.e. if $\quad ab' + a'b - 2hh' = 2\,(ab - h^2). \quad$ [Art. 377.]

13. The straight lines joining the intersections of the given conic with the circle $x^2 + y^2 = r^2$ are clearly

$$r^2(ax^2 + 2hxy + by^2) = x^2 + y^2,$$

i.e. $$x^2(ar^2 - 1) + 2hr^2xy + (br^2 - 1)y^2 = 0. \quad \text{[Art. 122.]}$$

These are conjugate diameters of the given conic if

$$a(br^2 - 1) - 2hr^2 . h + b(ar^2 - 1) = 0,$$

i.e. if $$r^2 = \frac{a+b}{2(ab-h^2)}.$$

Therefore the equation to the joining lines is

$$\frac{ax^2 + 2hxy + by^2}{ab - h^2} = \frac{2(x^2 + y^2)}{a+b}.$$

These lines, since they are conjugate diameters and also are of equal length r, are the equi-conjugates.

379. *Two conics, in general, intersect in four points, real or imaginary.*

For the general equation to two conics can be written in the form

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0,$$

and $$a'x^2 + 2x(h'y + g') + b'y^2 + 2f'y + c' = 0.$$

Eliminating x from these equations, we find that the result is an equation of the fourth degree in y, giving therefore four values, real or imaginary, for y. Also, by eliminating x^2 from these two equations, we see that there is only one value of x for each value of y. There are therefore only four points of intersection.

380. *Equation to any conic passing through the intersection of two given conics.*

Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots(1)$,

and $S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \ldots(2)$

be the equations to the two given conics.

Then $$S - \lambda S' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is the equation to any conic passing through the intersections of (1) and (2).

For, since S and S' are both of the second degree in x and y, the equation (3) is of the second degree, and hence represents a conic section.

Also, since (3) is satisfied when both S and S' are zero, it is satisfied by the points (real or imaginary) which are common to (1) and (2).

Hence (3) is a conic which passes through the intersections of (1) and (2).

381. *To find the equations to the straight lines passing through the intersections of two conics given by the general equations.*

As in the last article, the equation

$$(a - \lambda a')\, x^2 + 2\,(h - \lambda h')\, xy + (b - \lambda b')\, x^2 + 2\,(g - \lambda g')\, x$$
$$+ 2\,(f - \lambda f')\, y + (c - \lambda c') = 0 \ldots\ldots(1),$$

represents some conic through the intersections of the given conics.

Now, by Art. 116, (1) represents straight lines if

$$(a - \lambda a')\,(b - \lambda b')\,(c - \lambda c') + 2\,(f - \lambda f')\,(g - \lambda g')\,(h - \lambda h')$$
$$- (a - \lambda a')\,(f - \lambda f')^2 - (b - \lambda b')\,(g - \lambda g')^2 - (c - \lambda c')\,(h - \lambda h')^2$$
$$= 0 \ldots\ldots(2).$$

Now (2) is a cubic equation. The three values of λ found from it will, when substituted successively in (1), give the three pairs of straight lines which can be drawn through the (real or imaginary) intersections of the two conics.

Also, since a cubic equation always has one real root, one value of λ is real, and it could be shown that there can always be drawn one pair of real straight lines through the intersections of two conics.

382. *All conics which pass through the intersections of two rectangular hyperbolas are themselves rectangular hyperbolas.*

In this case, if $S=0$ and $S'=0$ be the two rectangular hyperbolas, we have

$$a+b=0, \text{ and } a'+b'=0. \quad \text{(Art. 358.)}$$

Hence, in the conic $S-\lambda S'=0$, the sum of the coefficients of x^2 and y^2

$$=(a-\lambda a')+(b-\lambda b')=(a+b)-\lambda(a'+b')=0.$$

Hence, the conic $S-\lambda S'=0$, *i.e.* any conic through the intersections of the two rectangular hyperbolas, is itself a rectangular hyperbola.

Cor. If two rectangular hyperbolas intersect in four points A, B, C, and D, the two straight lines AD and BC, which are a conic through the intersection of the two hyperbolas, must be a rectangular hyperbola. Hence AD and BC must be at right angles. Similarly, BD and CA, and CD and AB, must be at right angles. Hence D is the orthocentre of the triangle ABC.

Therefore, if two rectangular hyperbolas intersect in four points, each point is the orthocentre of the triangle formed by the other three.

383. *If $L=0$, $M=0$, $N=0$, and $R=0$ be the equations to the four sides of a quadrilateral taken in order, the equation to any conic passing through its angular points is*

$$LN=\lambda . MR \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For $L=0$ passes through one pair of its angular points and $N=0$ passes through the other pair. Hence $LN=0$ is the equation to a conic (viz. a pair of straight lines) passing through the four angular points.

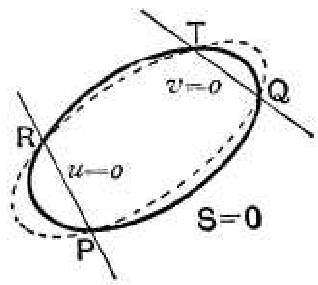
Similarly $MR = 0$ is the equation to another conic passing through the four points.

Hence $LN = \lambda . MR$ is the equation to any conic through the four points.

Geometrical meaning. Since L is proportional to the perpendicular from any point (x, y) upon the straight line $L = 0$, the relation (1) states that the product of the perpendiculars from any point of the curve upon the straight lines $L = 0$ and $N = 0$ is proportional to the product of the perpendiculars from the same point upon $M = 0$ and $R = 0$.

Hence *If a conic circumscribe a quadrilateral, the ratio of the product of the perpendiculars from any point P of the conic upon two opposite sides of the quadrilateral to the product of the perpendiculars from P upon the other two sides is the same for all positions of P.*

384. *Equations to the conic sections passing through the intersections of a conic and two given straight lines.*



Let $S = 0$ be the equation to the given conic.

Let $u = 0$ and $v = 0$ be the equations to the two given straight lines where

$$u \equiv ax + by + c,$$

and $$v \equiv a'x + b'y + c'.$$

Let the straight line $u = 0$ meet the conic $S = 0$ in the points P and R, and let $v = 0$ meet it in the points Q and T.

The equation to any conic which passes through the points P, Q, R, and T will be of the form

$$S = \lambda . u . v \qquad (1).$$

For (1) is satisfied by the coordinates of any point which lies both on $S = 0$ and on $u = 0$; for its coordinates on being substituted in (1) make both its members zero.

But the points P and R are the only points which lie both on $S = 0$ and on $u = 0$.

The equation (1) therefore denotes a conic passing through P and R.

Similarly it goes through the intersections of $S = 0$ and $v = 0$, *i.e.* through the points Q and T.

Thus (1) represents some conic going through the four points P, Q, R, and T.

Also (1) represents any conic going through these four points. For the quantity λ may be so chosen that it shall go through any fifth point, or to make it satisfy any fifth condition; also five conditions completely determine a conic section.

Ex. *Find the equation to the conic which passes through the point* (1, 1) *and also through the intersections of the conic*

$$x^2+2xy+5y^2-7x-8y+6=0$$

with the straight lines $2x-y-5=0$ *and* $3x+y-11=0$. *Find also the parabolas passing through the same points.*

The equation to the required conic must by the last article be of the form

$$x^2+2xy+5y^2-7x-8y+6=\lambda(2x-y-5)(3x+y-11) \ldots (1).$$

This passes through the point (1, 1) if

$$1+2+5-7-8+6=\lambda(2-1-5)(3+1-11), \text{ i.e. if } \lambda=-\tfrac{1}{28}.$$

The required equation then becomes

$$28(x^2+2xy+5y^2-7x-8y+6)+(2x-y-5)(3x+y-11)=0,$$

i.e. $$34x^2+55xy+139y^2-233x-218y+223=0.$$

The equation to the required parabola will also be of the form (1), *i.e.*

$$x^2(1-6\lambda)+xy(2+\lambda)+y^2(5+\lambda)-x(7-37\lambda)-y(8+6\lambda)+6-55\lambda=0.$$

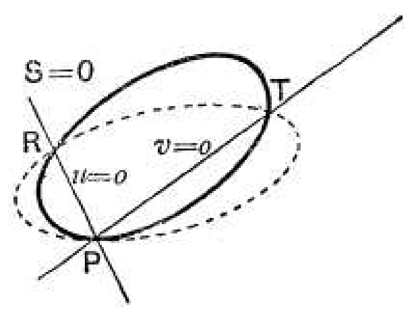
This is a parabola (Art. 357) if $(2+\lambda)^2=4(1-6\lambda)(5+\lambda)$,

i.e. if $$\lambda=\tfrac{1}{5}[-12\pm4\sqrt{10}].$$

Substituting these values in (1), we have the required equations.

385. Particular cases of the equation $S=\lambda uv$.

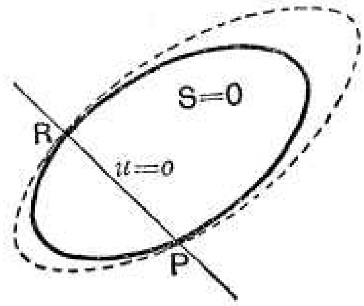
I. Let $u=0$ and $v=0$ intersect on the curve, *i.e.* in the figure of Art. 384 let the points P and Q coincide.



The conic $S=\lambda uv$ then goes through two coincident points at P and therefore touches the original conic at P as in the figure.

II. Let $u=0$ and $v=0$ coincide, so that $v=u$.

In this case the point T also moves up to coincidence with R and the second conic touches the original conic at both the points P and R.



The equation to the second conic now becomes $S=\lambda u^2$.

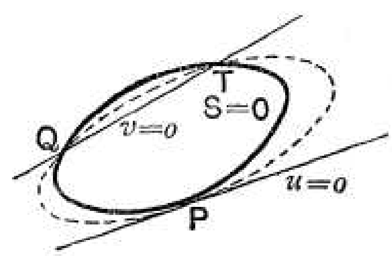
When a conic touches a second conic at each of two points, the two conics are said to have **double contact** with one another.

The two conics $S=\lambda u^2$ and $S=0$ therefore have double contact with one another, the straight line $u=0$ passing through the two points of contact.

As a particular case we see that if $u=0$, $v=0$, and $w=0$ be the equations to three straight lines then the equation $vw=\lambda u^2$ represents a conic touching the conic $vw=0$ where $u=0$ meets it, *i.e.* it is a conic to which $v=0$ and $w=0$ are tangents and $u=0$ is the chord of contact.

III. Let $u=0$ be a tangent to the original conic.

In this case the two points P and R coincide, and the conic $S=\lambda uv$ touches $S=0$ where $u=0$ touches it, and $v=0$ is the equation to the straight line joining the other points of intersection of the two conics.



If, in addition, $v=0$ goes through the point of contact of $u=0$, we have the equation to a conic which goes through three coincident points at P, the point of contact of $u=0$; also the straight line joining P to the other point of intersection of the two conics is $v=0$.

IV. Finally, let $v=0$ and $u=0$ coincide and be tangents at P. The equation $S=\lambda u^2$ now represents a conic section passing through four coincident points at the point where $u=0$ touches $S=0$.

386. Line at infinity. We have shewn, in Art. 60, that the straight line, whose equation is

$$0 . x + 0 . y + C = 0,$$

is altogether at an infinite distance. This straight line is called The Line at Infinity. Its equation may for brevity be written in the form $C = 0$.

We can shew that parallel lines meet on the line at infinity.

For the equations to any two parallel straight lines are

$$Ax + By + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$Ax + By + C' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now (2) may be written in the form

$$Ax + By + C + \frac{C' - C}{C}(0 . x + 0 . y + C) = 0,$$

and hence, by Art. 97, we see that it passes through the intersection of (1) and the straight line

$$0 . x + 0 . y + C = 0.$$

Hence (1), (2), and the line at infinity meet in a point.

387. *Geometrical meaning of the equation*

$$S = \lambda u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where λ *is a constant, and* $u = 0$ *is the equation of a straight line.*

The equation (1) can be written in the form

$$S = \lambda u \times (0 . x + 0 . y + 1),$$

and hence, by Art. 384, represents a conic passing through the intersection of the conic $S = 0$ with the straight lines

$$u = 0 \text{ and } 0 . x + 0 . y + 1 = 0.$$

Hence (1) passes through the intersection of $S = 0$ with the line at infinity.

Since $S = 0$ and $S = \lambda u$ have the same intersections with the line at infinity, it follows that these two conics have their asymptotes in the same direction.

Particular Case. Let

$$S \equiv x^2 + y^2 - a^2,$$

so that $S = 0$ represents a circle.

Any other circle is

$$x^2 + y^2 - 2gx - 2fy + c = 0,$$

i.e. $$x^2 + y^2 - a^2 = 2gx + 2fy - a^2 - c,$$

so that its equation is of the form $S = \lambda u$.

It therefore follows that any two circles must be looked upon as intersecting the line at infinity in the same two (imaginary) points. These imaginary points are called the Circular Points at Infinity.

388. *Geometrical meaning of the equation $S = \lambda$, where λ is a constant.*

This equation can be written in the form

$$S = \lambda (0 . x + 0 . y + 1)^2,$$

and therefore, by Art. 385, has double contact with $S = 0$ where the straight line $0 . x + 0 . y + 1 = 0$ meets it, *i.e.* the tangents to the two conics at the points where they meet the line at infinity are the same.

The conics $S = 0$ and $S = \lambda$ therefore have the same (real or imaginary) asymptotes.

Particular Case. Let $S = 0$ denote a circle. Then $S = \lambda$ (being an equation which differs from $S = 0$ only in its constant term) represents a concentric circle.

Two concentric circles must therefore be looked upon as touching one another at the imaginary points where they meet the Line at Infinity.

Two concentric circles thus have double contact at the Circular Points at Infinity.

EXAMPLES XLIII

1. What is the geometrical meaning of the equations $S=\lambda . T$, and $S=u^2+ku$, where $S=0$ is the equation of a conic, $T=0$ is the equation of a tangent to it, and $u=0$ is the equation of any straight line?

2. If the major axes of two conics be parallel, prove that the four points in which they meet are concyclic.

3. Prove that in general two parabolas can be drawn to pass through the intersections of the conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and $$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0,$$

and that their axes are at right angles if $h(a'-b')=h'(a-b)$.

4. Through the extremities of two focal chords of an ellipse a conic is described; if this conic pass through the centre of the ellipse, prove that it will cut the major axis in another fixed point.

5. Through the extremities of a normal chord of an ellipse a circle is drawn such that its other common chord passes through the centre of the ellipse. Prove that the locus of the intersection of these common chords is an ellipse similar to the given ellipse. If the eccentricity of the given ellipse be $\sqrt{2(\sqrt{2}-1)}$, prove that the two ellipses are equal.

6. If two rectangular hyperbolas intersect in four points A, B, C, and D, prove that the circles described on AB and CD as diameters cut one another orthogonally.

7. A circle is drawn through the centre of the rectangular hyperbola $xy=c^2$ to touch the curve and meet it again in two points; prove that the locus of the feet of the perpendicular let fall from the centre upon the common chord is the hyperbola $4xy=c^2$.

8. If a circle touch an ellipse and pass through its centre, prove that the rectangle contained by the perpendiculars from the centre of the ellipse upon the common tangent and the common chord is constant for all points of contact.

9. From a point T whose coordinates are (x', y') a pair of tangents TP and TQ are drawn to the parabola $y^2=4ax$; prove that the line joining the other pair of points in which the circumcircle of the triangle TPQ meets the parabola is the polar of the point $(2a-x', y')$, and hence that, if the circle touch the parabola, the line PQ touches an equal parabola.

10. Prove that the equation to the circle, having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus rectum, is

$$x^2+y^2-2ae^3x=a^2(1-e^2-e^4).$$

11. Two circles have double contact with a conic, their chords of contact being parallel. Prove that the radical axis of the two circles is midway between the two chords of contact.

12. If a circle and an ellipse have double contact with one another, prove that the length of the tangent drawn from any point of the ellipse to the circle varies as the distance of that point from the chord of contact.

13. Two conics, A and B, have double contact with a third conic C. Prove that two of the common chords of A and B, and their chords of contact with C, meet in a point.

14. Prove that the general equation to the ellipse, having double contact with the circle $x^2+y^2=a^2$ and touching the axis of x at the origin, is $$c^2x^2+(a^2+c^2)y^2-2a^2cy=0.$$

15. A rectangular hyperbola has double contact with a fixed central conic. If the chord of contact always passes through a fixed point, prove that the locus of the centre of the hyperbola is a circle passing through the centre of the fixed conic.

16. A rectangular hyperbola has double contact with a parabola; prove that the centre of the hyperbola and the pole of the chord of contact are equidistant from the directrix of the parabola.

ANSWERS

1. A conic touching $S=0$ where $T=0$ touches it and having its asymptotes parallel to those of $S=0$.

   A conic such that the two parallel straight lines $u=0$ and $u+k=0$ pass through its intersections with $S=0$.

SOLUTIONS/HINTS

1. The conic (1) passes through the intersections of $S=0$ and $T=0$, and these intersections are coincident since $T=0$ is a tangent. Hence it touches $S=0$ at the point where $T=0$ touches it, and since the terms of the second degree are the same, the two conics have parallel asymptotes.

The conic (2) passes through the intersection of the conic $S=0$ with the parallel straight lines $u=0$ and

$$u+k=0.$$

2. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and $\frac{(x-c)^2}{\alpha^2}+\frac{(y-d)^2}{\beta^2}=1$,

be the equations of the two conics.

Any conic through their intersections is

$$\frac{(x-c)^2}{\alpha^2}+\frac{(y-d)^2}{\beta^2}-1+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0.$$

Since there is no term in xy, this conic will be a circle if the coefficients of x^2 and y^2 are equal, that is if

$$\frac{1}{\alpha^2}+\frac{\lambda}{a^2}=\frac{1}{\beta^2}+\frac{\lambda}{b^2},$$

and this gives a real value for λ, so that a circle can be drawn through their intersections.

3. The conic $S-\lambda S'=0$ of Art. 380 is a parabola if

$$(a-\lambda a')(b-\lambda b')=(h-\lambda h')^2,$$

that is, if $\lambda^2(a'b'-h'^2)+\lambda(2hh'-ab'-a'b)+ab-h^2=0$.

Since this is a quadratic, there will be two values of λ that will make the above conic a parabola.

[If two parabolas have their axes at right angles, their equations can clearly be expressed in the form

$$x^2=Ax+By+C \text{ and } y^2=A'x+B'y+C',$$

so that their four points of intersection lie on the circle

$$x^2+y^2=(A+A')x+(B+B')y+(C+C').]$$

The conic $S-\lambda S'=0$ must therefore, for some value of λ, be a circle. The conditions for this are

$$a-\lambda a'=b-\lambda b', \text{ and } h-\lambda h'=0.$$

On eliminating λ, we have $h(a'-b')=h'(a-b)$.

4. Let $y=m_1(x-ae)$ and $y=m_2(x-ae)$

be the equations of any two focal chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Any conic going through the intersections of these chords and the ellipse is

$$\{y - m_1(x - ae)\}\{y - m_2(x - ae)\} - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0.$$

If it passes through (0, 0), then $m_1 m_2 a^2 e^2 + \lambda = 0$.

Putting $y = 0$, the points in which the curve cuts the major axis are given by

$$m_1 m_2 (x - ae)^2 - \lambda\left(\frac{x^2}{a^2} - 1\right) = 0,$$

i.e. by $(x - ae)^2 + e^2(x^2 - a^2) = 0$, *i.e.* by $x^2 - 2aex + e^2x^2 = 0$.

Therefore the other fixed point is $\left(\frac{2ae}{1 + e^2}, 0\right)$.

5. Let $y = mx$ be the other common chord.

A conic through their intersections is

$$(ax \sec\phi - by \operatorname{cosec}\phi - a^2 + b^2)(y - mx) - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

which will be a circle if $mb \operatorname{cosec}\phi + a \sec\phi = 0$.

Hence we have to eliminate ϕ between

$$ax \sin\phi - by \cos\phi = (a^2 - b^2)\sin\phi\cos\phi,$$

and
$$\frac{\cos\phi}{ax} = -\frac{\sin\phi}{by} = \frac{1}{\sqrt{a^2x^2 + b^2y^2}}.$$

$$\therefore\ \sqrt{a^2x^2 + b^2y^2}\,(-2abxy) = (a^2 - b^2)(-axby).$$

$$\therefore\ a^2x^2 + b^2y^2 = \frac{(a^2 - b^2)^2}{4},$$

which is the equation of a similar ellipse. The ellipses are equal if $\frac{(a^2 - b^2)^2}{4b^2} = a^2$,

i.e. if $e^4 = 4(1 - e^2)$, *i.e.* if $e^2 = 2(\sqrt{2} - 1)$.

6. By Art. 382, Cor., D is the orthocentre of the triangle ABC. Let AD, BD, CD meet the opposite sides, to which they are perpendicular, in X, Y, Z. Then the two circles spoken of go through X and Y. The angle the tangent to the first at Y makes with YA = the alternate angle ABY, and the angle the tangent to the second at Y makes with $YB =$ the alternate angle YDC.

Hence the angle between the two tangents at Y

$$= \angle ABY + 90^\circ - \angle YDC = \text{a right angle.} \quad \text{Hence, etc.}$$

7. Let $x\cos\alpha + y\sin\alpha - p = 0$ be the equation of the common chord; and (x', y') the point where the circle touches the hyperbola, so that the tangent to the latter at it is $\frac{x}{x'} + \frac{y}{y'} = 2$.

The conic

$$\left(\frac{x}{x'} + \frac{y}{y'} - 2\right)(x\cos\alpha + y\sin\alpha - p) + \lambda(xy - c^2) = 0$$

will pass through the centre if $2p - \lambda c^2 = 0$, and is a circle if

$$\frac{\cos\alpha}{y'} + \frac{\sin\alpha}{x'} + \frac{2p}{c^2} = 0, \text{ and } \frac{\cos\alpha}{x'} = \frac{\sin\alpha}{y'} = \frac{\sqrt{\cos\alpha\sin\alpha}}{c}.$$

Substituting for x', y' and putting $x = p\cos\alpha$, $y = p\sin\alpha$ we obtain $4xy = c^2$ as the required locus.

8. Let $x\cos\alpha + y\sin\alpha = p$ be the equation of the common chord.

The conic

$$\left(\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1\right)(x\cos\alpha + y\sin\alpha - p) + p\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

passes through the centre and is a circle if

$$\frac{\cos\alpha\cos\phi}{a} + \frac{p}{a^2} = \frac{\sin\alpha\sin\phi}{b} + \frac{p}{b^2},$$

and

$$\frac{\cos\alpha\sin\phi}{b} + \frac{\sin\alpha\cos\phi}{a} = 0,$$

whence eliminating α, $a^2\sin^2\phi + b^2\cos^2\phi = \frac{p^2(a^2 - b^2)^2}{a^2b^2}$.

The required rectangle $= \frac{pab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} = \frac{a^2b^2}{a^2 - b^2}$.

9. Let $cx+dy=1$ be the equation of the other common chord.

The conic

$$\{yy'-2a(x+x')\}\{cx+dy-1\}-\lambda(y^2-4ax)=0$$

passes through (x', y') if $\lambda=cx'+dy'-1$, and is a circle if

$$cy'-2ad=0,$$

and $$-2ac=dy'-\lambda=dy'-(cx'+dy'-1)=1-cx'.$$

$$\therefore\ c=\frac{1}{x'-2a} \text{ and } d=\frac{y'}{2a(x'-2a)}.$$

The equation of the common chord becomes

$$x+\frac{yy'}{2a}=x'-2a, \text{ i.e. } yy'+2a(x+2a-x')=0,$$

which is the polar of $(2a-x', -y')$.

If the circle touch the parabola, this common chord must touch the parabola and hence its pole must lie on the curve; $\therefore\ y'^2=4a(2a-x')$.

Substitute for x' in the equation of PQ and it becomes

$$y=\frac{2a}{y'}(x+2a)-\frac{a}{2a/y},$$

which shows that it always touches the parabola

$$y^2=-4a(x+2a).$$

10. The conic $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\lambda(x-ae)^2=0$$

is a circle if $$\lambda=\frac{1}{a^2}-\frac{1}{b^2}.$$

Substitute for λ, and it becomes

$$b^2x^2+a^2y^2-a^2b^2=-a^2e^2(x^2-2aex+a^2e^2),$$

i.e. $$x^2+y^2-2ae^3x=a^2(1-e^2-e^4).$$

11. Let $x\cos\alpha+y\sin\alpha-p=0$ be one of the chords of contact. The equation to the circle is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x\cos\alpha+y\sin\alpha-p)^2=0.$$

Since this is a circle, the coefficient of xy is zero.

$$\therefore\ \lambda\cos\alpha\sin\alpha=0.$$

Therefore $\alpha=0$ or $\frac{\pi}{2}$, showing that the chord of contact must be either parallel to the major or the minor axis.

Taking the first case, the circle is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-c)^2=0,$$

(where $x=c$ is the common chord) and $\frac{1}{a^2}+\lambda=\frac{1}{b^2}$. Hence the circle is

$$x^2+y^2-2cx\frac{a^2-b^2}{a^2}+b^2+\frac{a^2-b^2}{a^2}c^2=0;$$

so the other is $x^2+y^2-2c'x\frac{a^2-b^2}{a^2}+b^2+\frac{a^2-b^2}{a^2}c'^2=0.$

By subtraction, the equation to the radical axis is

$$x=\frac{c+c'}{2}.$$

12. As in the last example, the circle

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(x-c)^2=0$$

has double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the square of the tangent from $(a\cos\phi, b\sin\phi)$ to the circle is

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(a\cos\phi-c)^2,$$

which $\qquad \propto (a\cos\phi-c)^2.$

13. Let $ax^2+by^2=1$ be the equation of C, and let

$$p_1x+q_1y+r_1=0,\ p_2x+q_2y+r_2=0$$

be the equations of the chords of contact.

The equations of A and B are

$$ax^2 + by^2 - 1 + \lambda_1^2 (p_1 x + q_1 y + r_1)^2 = 0,$$

and
$$ax^2 + by^2 - 1 + \lambda_2^2 (p_2 x + q_2 y + r_2)^2 = 0.$$

By subtraction, two of the common chords are

$$\lambda_1 (p_1 x + q_1 y + r_1) = \pm \lambda_2 (p_2 x + q_2 y + r_2).$$

14. Assuming that the chord of contact is parallel to the axis of x, let its equation be $y = c$.

The required equation of the conic is

$$x^2 + y^2 - a^2 + \lambda (y - c)^2 = 0.$$

Since it touches the axis of x at the origin, $\therefore\ \lambda c^2 = a^2$.

Hence the equation becomes

$$c^2 (x^2 + y^2 - a^2) + a^2 (y^2 - 2cy + c^2) = 0,$$

i.e.
$$c^2 x^2 + y^2 (a^2 + c^2) - 2a^2 cy = 0.$$

15. Let $ax^2 + by^2 - 1 = 0$ be the equation of the fixed central conic.

The conic $ax^2 + by^2 - 1 + \lambda (px + qy - 1)^2 = 0$, is a rectangular hyperbola if

$$a + b + \lambda (p^2 + q^2) = 0. \quad \text{.................(i)}$$

The centre is given by

$$-ax = \lambda p (px + qy - 1), \quad \text{..............(ii)}$$

and
$$-by = \lambda q (px + qy - 1). \quad \text{...............(iii)}$$

If (h, k) be the fixed point, $ph + qk = 1$. ....(iv)

From (ii) and (iii)

$$-\lambda (px + qy - 1) = \frac{ax}{p} = \frac{by}{q} = axh + byk, \text{ from (iv)}.$$

Substitute for p and q in (i) and (ii) ;

$$\therefore\ (axh + byk)^2 (a + b) + \lambda (a^2 x^2 + b^2 y^2) = 0,$$

and
$$(axh + byk)^2 + \lambda (ax^2 + by^2 - axh - byk) = 0.$$

Eliminating λ,

$$\therefore\ (a + b) \{ax (x - h) + by (y - k)\} = a^2 x^2 + b^2 y^2,$$

which is a circle passing through (0, 0).

16. The conic $(lx+my-1)^2+\lambda(y^2-4ax)=0$ is a rectangular hyperbola if $l^2+m^2+\lambda=0$.

Substituting for λ, the equation becomes

$$l^2x^2+2lmxy-l^2y^2+2x\{2a(l^2+m^2)-l\}-2my+1=0.$$

The centre is given by

$$l^2x+lmy+2a(l^2+m^2)-l=0, \text{ and } lmx-l^2y-m=0.$$

$$\therefore\ x=\frac{1}{l}-2a.$$

If (x', y') be the pole of the chord, then

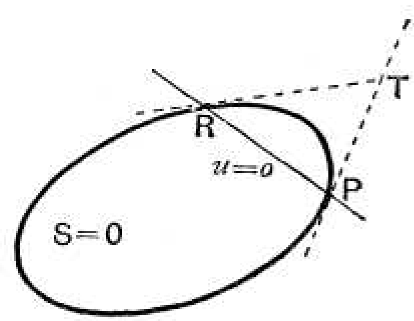
$$x'=-\frac{1}{l}.\quad \text{[Art. 218.]}$$

Therefore the sum of the abscissae of the centre and pole $=-2a=$ twice the abscissa of the foot of the directrix. Hence they are equidistant from the directrix.

389. *To find the equation of the pair of tangents that can be drawn from any point (x', y') to the general conic*

$$\phi(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let T be the given point (x', y'), and let P and R be the points where the tangents from T touch the conic.



The equation to PR is therefore $u=0$,

where $u\equiv(ax'+hy'+g)x+(hx'+by'+f)y+gx'+fy'+c$.

The equation to any conic which touches $S=0$ at both of the points P and R is

$$S=\lambda u^2,\quad \text{(Art. 385)},$$

i.e.

$$ax^2+2hxy+by^2+2gx+2fy+c$$
$$=\lambda[(ax'+hy'+g)x+(hx'+by'+f)y+gx'+fy'+c]^2 \quad \ldots\ldots(1).$$

Now the pair of straight lines TP and TR is a conic section which touches the given conic at P and R and which also goes through the point T.

Also we can only draw one conic to go through five points, viz. T, two points at P, and two points at R.

If then we find λ so that (1) goes through the point T, it must represent the two tangents TP and TR.

The equation (1) is satisfied by x' and y' if

$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$$
$$= \lambda \left[ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c\right]^2,$$

i.e. if $$\lambda = \frac{1}{\phi(x', y')}.$$

The required equation (1) then becomes

$$\phi(x', y')\left[ax^2 + 2hxy + by^2 + 2gx + 2fy + c\right]$$
$$= \left[(ax' + hy' + g)x + (hx' + by' + f)y + gx' + fy' + c\right]^2,$$

i.e. $$\boldsymbol{\phi(x, y) \times \phi(x', y') = u^2},$$

where $u = 0$ is the equation to the chord of contact.

390. *Director circle of a conic given by the general equation of the second degree.*

The equation to the two tangents from (x', y') to the conic are, by the last article,

$$x^2\left[a\phi(x', y') - (ax' + hy' + g)^2\right]$$
$$+ 2xy\left[h\phi(x', y') - (ax' + hy' + g)(hx' + by' + f)\right]$$
$$+ y^2\left[b\phi(x', y') - (hx' + by' + f)^2\right] + \text{other terms} = 0 \ldots (1).$$

If (x', y') be a point on the director circle of the conic, the two tangents from it to the conic are at right angles.

Now (1) represents two straight lines at right angles if the sum of the coefficients of x^2 and y^2 in it be zero,

i.e. if $(a + b)\phi(x', y') - (ax' + hy' + g)^2 - (hx' + by' + f)^2 = 0.$

Hence the locus of the point (x', y') is

$$(a + b)(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$- (ax + hy + g)^2 - (hx + by + f)^2 = 0,$$

i.e. the circle whose equation is

$$(x^2 + y^2)(ab - h^2) + 2x(bg - fh) + 2y(af - gh)$$
$$+ c(a + b) - g^2 - f^2 = 0.$$

Cor. If the given conic be a parabola, then $ab = h^2$, and the locus becomes a straight line, viz. the directrix of the parabola. (Art. 211.)

391. The equation to the director circle may also be obtained in another manner. For it is a circle, whose centre is at the centre of the conic, and the square of whose radius is equal to the sum of the squares of the semi-axes of the conic.

The centre is, Art. 352, the point $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

Also, if the equation to the conic be reduced to the form

$$ax^2 + 2hxy + by^2 + c' = 0,$$

and if α and β be its semi-axes, we have, (Art. 364,)

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{a+b}{-c'}, \text{ and } \frac{1}{\alpha^2\beta^2} = \frac{ab - h^2}{c'^2},$$

so that, by division, $\alpha^2 + \beta^2 = \frac{-(a+b)c'}{ab - h^2}$.

The equation to the required circle is therefore

$$\left(x - \frac{hf - bg}{ab - h^2}\right)^2 + \left(y - \frac{gh - af}{ab - h^2}\right)^2 = -\frac{(a+b)c'}{ab - h^2}$$

$$= -\frac{(a+b)(abc + 2fgh - af^2 - bg^2 - ch^2)}{(ab - h^2)^2} \quad \text{(Art. 352).}$$

392. *The equation to the (imaginary) tangents drawn from the focus of a conic to touch the conic satisfies the analytical condition for being a circle.*

Take the focus of the conic as origin, and let the axis of x be perpendicular to its directrix, so that the equation to the latter may be written in the form $x + k = 0$.

The equation to the conic, e being its eccentricity, is therefore

$$x^2 + y^2 = e^2(x + k)^2,$$

i.e.

$$x^2(1 - e^2) + y^2 - 2e^2kx - e^2k^2 = 0.$$

The equation to the pair of tangents drawn from the origin is therefore, by Art. 389,

$$[x^2(1 - e^2) + y^2 - 2e^2kx - e^2k^2][-e^2k^2] = [-e^2kx - e^2k^2]^2,$$

i.e.

$$x^2(1 - e^2) + y^2 - 2e^2kx - e^2k^2 = -e^2[x + k]^2,$$

i.e.

$$x^2 + y^2 = 0 \quad \text{..........(1).}$$

Here the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero.

However the axes and origin of coordinates be changed, it follows, on making the substitutions of Art. 129, that in (1) the coefficients of x^2 and y^2 will still be equal and the coefficient of xy zero.

Hence, whatever be the conic and however its equation may be written, the equation to the tangents from the focus always satisfies the analytical conditions for being a circle.

393. *To find the foci of the conic given by the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let (x', y') be a focus. By the last article the equation to the pair of tangents drawn from it satisfies the conditions for being a circle.

The equation to the pair of tangents is

$$\phi(x', y')\left[ax^2 + 2hxy + by^2 + 2gx + 2fy + c\right]$$
$$= \left[x(ax' + hy' + g) + y(hx' + by' + f) + (gx' + fy' + c)\right]^2.$$

In this equation the coefficients of x^2 and y^2 must be equal and the coefficient of xy must be zero.

We therefore have

$$a\phi(x', y') - (ax' + hy' + g)^2 = b\phi(x', y') - (hx' + by' + f)^2,$$

and $$h\phi(x', y') = (ax' + hy' + g)(hx' + by' + f),$$

i.e.

$$\frac{(ax' + hy' + g)^2 - (hx' + by' + f)^2}{a - b} = \frac{(ax' + hy' + g)(hx' + by' + f)}{h}$$
$$= \phi(x', y') \;\ldots\ldots\ldots\ldots\ldots (4).$$

These equations, on being solved, give the foci.

Cor. Since the directrices are the polars of the foci, we easily obtain their equations.

394. The equations (4) of the previous article give, in general, four values for x' and four corresponding values for y'. Two of these would be found to be real and two imaginary.

In the case of the ellipse the two imaginary foci lie on the minor axis. That these imaginary foci exist follows from Art. 247, by writing the standard equation in the form

$$x^2 + \{y - \sqrt{b^2 - a^2}\}^2 = \frac{b^2 - a^2}{b^2}\left\{y - \frac{b^2}{\sqrt{b^2 - a^2}}\right\}^2.$$

This shews that the imaginary point $\{0, \sqrt{b^2-a^2}\}$ is a focus, the imaginary line $y - \dfrac{b^2}{\sqrt{b^2-a^2}}=0$ is a directrix, and that the corresponding eccentricity is the imaginary quantity $\sqrt{\dfrac{b^2-a^2}{b^2}}$.

Similarly for the hyperbola, except that, in this case, the eccentricity is real.

In the case of the parabola, two of the foci are at infinity and are imaginary, whilst a third is at infinity and is real.

395. Ex. 1. *Find the focus of the parabola*

$$16x^2-24xy+9y^2-80x-140y+100=0.$$

The focus is given by the equations

$$\frac{(16x'-12y'-40)^2-(-12x'+9y'-70)^2}{7}$$
$$=\frac{(16x'-12y'-40)(-12x'+9y'-70)}{-12}$$
$$=16x'^2-24x'y'+9y'^2-80x'-140y'+100\ldots\ldots\ldots\ldots(1).$$

The first pair of equation (1) give

$$12(16x'-12y'-40)^2+7(16x'-12y'-40)(-12x'+9y'-70)$$
$$-12(-12x'+9y'-70)^2=0,$$

i.e. $\{4(16x'-12y'-40)-3(-12x'+9y'-70)\}$
$$\times\{3(16x'-12y'-40)+4(-12x'+9y'-70)\}=0,$$

i.e. $$(100x'-75y'+50)\times(-400)=0,$$

so that $$y'=\frac{4x'+2}{3}.$$

We then have $$16x'-12y'-40=-48,$$

and $$-12x'+9y'-70=-64.$$

The second pair of equation (1) then gives

$$-\frac{48\times 64}{12}=x'(16x'-12y'-40)+y'(-12x'+9y'-70)-40x'-70y'+100$$
$$=-48x'-64y'-40x'-70y'+100$$
$$=-88x'-134y'+100,$$

i.e. $-256=-88x'-\dfrac{536x'+268}{3}+100,$

so that $x'=1$, and then $y'=2$.

The focus is therefore the point (1, 2).

In the case of a parabola, we may also find the equation to the directrix, by Art. 390, and then find the coordinates of the focus, which is the pole of the directrix.

Ex. 2. *Find the foci of the conic*

$$55x^2 - 30xy + 39y^2 - 40x - 24y - 464 = 0.$$

The foci are given by the equation

$$\frac{(55x' - 15y' - 20)^2 - (-15x' + 39y' - 12)^2}{16}$$
$$= \frac{(55x' - 15y' - 20)(-15x' + 39y' - 12)}{-15}$$
$$= 55x'^2 - 30x'y' + 39y'^2 - 40x' - 24y' - 464 \text{ ....(1).}$$

The first pair of equations (1) gives

$$15(55x' - 15y' - 20)^2 + 16(55x' - 15y' - 20)(-15x' + 39y' - 12)$$
$$- 15(-15x' + 39y' - 12)^2 = 0,$$

i.e. $$\{5(55x' - 15y' - 20) - 3(-15x' + 39y' - 12)\}$$
$$\{3(55x' - 15y' - 20) + 5(-15x' + 39y' - 12)\} = 0,$$

i.e. $$(5x' - 3y' - 1)(3x' + 5y' - 4) = 0.$$

$$\therefore \ y' = \frac{5x' - 1}{3} \text{ ....(2),}$$

or $$y' = -\frac{3x' - 4}{5} \text{ .... (3).}$$

Substituting this first value of y in the second pair of equation (1), we obtain

$$-25(2x' - 1)^2 = \frac{340x'^2 - 340x' - 1355}{3},$$

giving $x' = 2$ or -1. Hence from (2) $y' = 3$ or -2.

On substituting the second value of y' in the same pair of equation (1), we finally have

$$2x'^2 - 2x' + 13 = 0,$$

the roots of which are imaginary.

We should thus obtain two imaginary foci which would be found to lie on the minor axis of the conic section. The real foci are therefore the points (2, 3) and (–1, –2).

396. Equation to the axes of the general conic.

By Art. 393, the equation

$$\frac{(ax + hy + g)^2 - (hx + by + f)^2}{a - b} = \frac{(ax + hy + g)(hx + by + f)}{h} \text{ ....(1)}$$

represents some conic passing through the foci.

But, since it could be solved as a quadratic equation to give $\frac{ax+hy+g}{hx+by+f}$, it represents two straight lines.

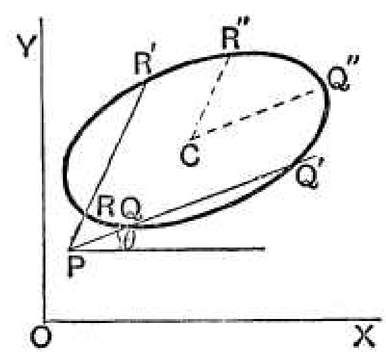
The equation (1) therefore represents the axes of the general conic.

397. *To find the length of the straight lines drawn through a given point in a given direction to meet a given conic.*

Let the equation to the conic be

$$\phi(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots(1).$$

Let P be any point (x', y'), and through it let there be drawn a straight line at an angle θ with the axis of x to meet the curve in Q and Q'.



The coordinates of any point on this line distant r from P are

$x' + r\cos\theta$ and $y' + r\sin\theta$. (Art. 86.)

If this point be on (1), we have

$$a(x'+r\cos\theta)^2 + 2h(x'+r\cos\theta)(y'+r\sin\theta) + b(y'+r\sin\theta)^2 + 2g(x'+r\cos\theta) + 2f(y'+r\sin\theta) + c = 0,$$

i.e.

$$r^2[a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta] + 2r[(ax'+hy'+g)\cos\theta + (hx'+by'+f)\sin\theta] + \phi(x', y') = 0 \quad \ldots\ldots\ldots\ldots(2).$$

For any given value of θ this is a quadratic equation in r, and therefore for any straight line drawn at an inclination θ it gives the values of PQ and PQ'.

If the two values of r given by equation (2) be of opposite sign, the points Q and Q' lie on opposite sides of P.

If P be on the curve, then $\phi(x', y')$ is zero and one value of r obtained from (2) is zero.

398. *If two chords PQQ' and PRR' be drawn in given directions through any point P to meet the curve in Q, Q' and R, R' respectively, the ratio of the rectangle $PQ \,.\, PQ'$ to the rectangle $PR \,.\, PR'$ is the same for all points, and is therefore equal to the ratio of the squares of the diameters of the conic which are drawn in the given directions.*

The values of PQ and PQ' are given by the equation of the last article, and therefore

$$PQ \,.\, PQ' = \text{product of the roots}$$

$$= \frac{\phi(x', y')}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta} \dots(1).$$

So, if PRR' be drawn at an angle θ' to the axis, we have

$$PR \,.\, PR' = \frac{\phi(x', y')}{a\cos^2\theta' + 2h\cos\theta'\sin\theta' + b\sin^2\theta'} \dots(2).$$

On dividing (1) by (2), we have

$$\frac{PQ \,.\, PQ'}{PR \,.\, PR'} = \frac{a\cos^2\theta' + 2h\cos\theta'\sin\theta' + b\sin^2\theta'}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}.$$

The right-hand member of this equation does not contain x' or y', *i.e.* it does not depend on the position of P but only on the directions θ and θ'.

The quantity $\dfrac{PQ \,.\, PQ'}{PR \,.\, PR'}$ is therefore the same for all positions of P.

In the particular case when P is at the centre of the conic this ratio becomes $\dfrac{CQ''^2}{CR''^2}$, where C is the centre and CQ'' and CR'' are parallel to the two given directions.

Cor. If Q and Q' coincide, and also R and R', the two lines PQQ' and PRR' become the tangents from P, and the above relation then gives

$$\frac{PQ^2}{PR^2} = \frac{CQ''^2}{CR''^2}, \quad i.e. \quad \frac{PQ}{PR} = \frac{CQ''}{CR''}.$$

Hence, *If two tangents be drawn from a point to a conic, their lengths are to one another in the ratio of the parallel semi-diameters of the conic.*

399. *If PQQ' and $P_1Q_1Q_1'$ be two chords drawn in parallel directions from two points P and P_1 to meet a conic in Q and Q', and Q_1 and Q_1', respectively, then the ratio of the rectangles $PQ.PQ'$ and $P_1Q_1.P_1Q_1'$ is independent of the direction of the chords.*

For, if P and P_1 be respectively the points (x', y') and (x'', y''), and θ be the angle that each chord makes with the axis, we have, as in the last article,

$$PQ.PQ' = \frac{\phi(x', y')}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta},$$

and $$P_1Q_1.P_1Q_1' = \frac{\phi(x'', y'')}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta},$$

so that $$\frac{PQ.PQ'}{P_1Q_1.P_1Q_1'} = \frac{\phi(x', y')}{\phi(x'', y'')}.$$

400. *If a circle and a conic section cut one another in four points, the straight line joining one pair of points of intersection and the straight line joining the other pair are equally inclined to the axis of the conic.*

For (Fig. Art. 397) let the circle and conic intersect in the four points Q, Q' and R, R' and let QQ' and RR' meet in P.

Then $$\frac{PQ.PQ'}{PR.PR'} = \frac{CQ''^2}{CR''^2} \quad \text{(Art. 398)}.$$

But, since Q, Q', R, and R' are four points on a circle, we have

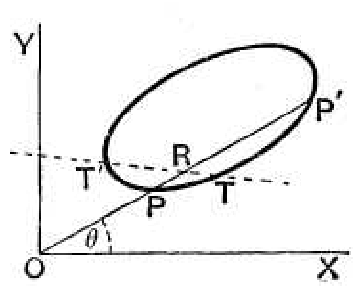
$$PQ.PQ' = PR.PR'. \quad \text{[Euc. III. 36, Cor.]}$$

$$\therefore\ CQ'' = CR''.$$

Also in any conic equal radii from the centre are equally inclined to the axis of the conic.

Hence CQ'' and CR'', and therefore PQQ' and PRR', are equally inclined to the axis of the conic.

401. *To shew that any chord of a conic is cut harmonically by the curve, any point on the chord, and the polar of this point with respect to the conic.*



Take the point as origin, and let the equation to the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots(1),$$

or, in polar coordinates,

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0,$$

i.e.

$$c \cdot \frac{1}{r^2} + 2 \cdot \frac{1}{r} \cdot (g\cos\theta + f\sin\theta)$$
$$+ a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta = 0.$$

Hence, if the chord OPP' be drawn at an angle θ to OX, we have

$$\frac{1}{OP} + \frac{1}{OP'} = \text{sum of the roots of this equation in } \frac{1}{r}$$
$$= -2 \cdot \frac{g\cos\theta + f\sin\theta}{c}.$$

Let R be a point on this chord such that

$$\frac{2}{OR} = \frac{1}{OP} + \frac{1}{OP'}.$$

Then, if $OR = \rho$, we have

$$\frac{2}{\rho} = -2\frac{g\cos\theta + f\sin\theta}{c},$$

so that the locus of R is

$$g \cdot \rho\cos\theta + f \cdot \rho\sin\theta + c = 0,$$

or, in Cartesian coordinates,

$$gx + fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But (2) is the polar of the origin with respect to the conic (1), so that the locus of R is the polar of O.

The straight line PP' is therefore cut harmonically by O and the point in which it cuts the polar of O.

Ex. *Through any point O is drawn a straight line to cut a conic in P and P′ and on it is taken a point R such that OR is* (1) *the arithmetic mean, and* (2) *the geometric mean, between OP and OP′. Find in each case the locus of R.*

Using the same notation as in the last article, we have

$$OP + OP' = -2\,\frac{g\cos\theta + f\sin\theta}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta},$$

and $$OP \cdot OP' = \frac{c}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}.$$

(1) If R be the point (ρ, θ) we have

$$\rho = \tfrac{1}{2}(OP + OP') = -\frac{g\cos\theta + f\sin\theta}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta},$$

i.e. $$a\rho\cos^2\theta + 2h\rho\cos\theta\sin\theta + b\rho\sin^2\theta + g\cos\theta + f\sin\theta = 0,$$

i.e., in Cartesian coordinates,

$$ax^2 + 2hxy + by^2 + gx + fy = 0.$$

The locus is therefore a conic passing through O and the intersection of the conic and the polar of O, *i.e.* through the points T and T', and having its asymptotes parallel to those of the given conic.

(2) If R be the point (ρ, θ), we have in this case

$$\rho^2 = OP \cdot OP' = \frac{c}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta},$$

i.e. $$a\rho^2\cos^2\theta + 2h\rho^2\cos\theta\sin\theta + b\rho^2\sin^2\theta = c,$$

i.e. $$ax^2 + 2hxy + by^2 = c.$$

The locus is therefore a conic, having its centre at O and passing through T and T', and having its asymptotes parallel to those of the given conic.

402. *To find the locus of the middle points of parallel chords of a conic.* [Cf. Art. 376.]

The lengths of the segments of the chord drawn through the point (x', y') at an angle θ to the axis of x is given by equation (2) of Art. 397.

If (x', y') be the middle point of the chord the roots of this equation are equal in magnitude but opposite in sign, so that their algebraic sum is zero.

The coefficient of r in this equation is therefore zero, so that

$$(ax' + hy' + g)\cos\theta + (hx' + by' + f)\sin\theta = 0.$$

The locus of the middle point of chords inclined at an angle θ to the axis of x is therefore the straight line

$$(ax + hy + g) + (hx + by + f)\tan\theta = 0.$$

Hence the locus of the middle points of chords parallel to the line $y = mx$ is

$$(ax + hy + g) + (hx + by + f)\,m = 0,$$

i.e. $$x(a + hm) + (h + bm)\,y + g + fm = 0.$$

This is parallel to the line $y = m'x$ if

$$m' = -\frac{a + hm}{h + bm},$$

i.e. if $$a + h(m + m') + bmm' = 0.$$

This is therefore the condition that $y = mx$ and $y = m'x$ should be parallel to conjugate diameters.

403. *Equation to the pair of tangents drawn from a given point (x', y') to a given conic.* [Cf. Art. 389.]

If a straight line be drawn through (x', y'), the point P, to meet the conic in Q and Q', the lengths of PQ and PQ' are given by the equation

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)$$
$$+2r[(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta]+\phi(x', y')=0.$$

The roots of this equation are equal, *i.e.* the corresponding lines touch the conic, if

$$(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)\times\phi(x', y')$$
$$=[(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta]^2,$$

i.e. if $(a+2h\tan\theta+b\tan^2\theta)\times\phi(x', y')$
$$=[(ax'+hy'+g)+(hx'+by'+f)\tan\theta]^2 \ldots(1).$$

The roots of this equation give the corresponding directions of the tangents through P.

Also the equation to the line through P inclined at an angle θ to the axis of x is

$$\frac{y-y'}{x-x'}=\tan\theta \ldots\ldots\ldots\ldots(2).$$

If we substitute for $\tan\theta$ in (1) from (2) we shall get the equation to the pair of tangents from P.

On substitution we have

$$\{a(x-x')^2+2h(x-x')(y-y')+b(y-y')^2\}\phi(x', y')$$
$$=[(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')]^2.$$

This equation reduces to the form of Art. 389.

EXAMPLES XLIV

1. Two tangents are drawn to an ellipse from a point P; if the points in which these tangents meet the axes of the ellipse be concyclic, prove that the locus of P is a rectangular hyperbola.

2. A pair of tangents to the conic $Ax^2+By^2=1$ intercept a constant distance $2k$ on the axis of x; prove that the locus of their point of intersection is the curve

$$By^2(Ax^2+By^2-1)=Ak^2(By^2-1)^2.$$

3. Pairs of tangents are drawn to the conic $\alpha x^2+\beta y^2=1$ so as to be always parallel to conjugate diameters of the conic

$$ax^2+hxy+by^2=1;$$

shew that the locus of their point of intersection is the conic

$$ax^2+2hxy+by^2=\frac{a}{\alpha}+\frac{b}{\beta}.$$

4. Prove that the director circles of all conics which touch two given straight lines at given points have a common radical axis.

5. A parabola circumscribes a right-angled triangle. Taking its sides as the axes of coordinates, prove that the locus of the foot of the perpendicular from the right angle upon the directrix is the curve whose equation is

$$2xy(x^2+y^2)(hy+kx)+h^2y^4+k^2x^4=0,$$

and that the axis is one of the family of straight lines

$$y=mx-\frac{m^3h-k}{1+m^2},$$

where m is an arbitrary parameter and $2h$ and $2k$ are the sides of the triangle.

Find the foci of the curves

6. $300x^2+320xy+144y^2-1220x-768y+199=0.$

7. $16x^2-24xy+9y^2+28x+14y+21=0.$

8. $144x^2-120xy+25y^2+67x-42y+13=0.$

9. $x^2-6xy+y^2-10x-10y-19=0$ and also its directrices.

10. Prove that the foci of the conic

$$ax^2+2hxy+by^2=1$$

are given by the equations

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}=\frac{1}{a^2-b^2}.$$

11. Prove that the locus of the foci of all conics which touch the four lines $x=\pm a$ and $y=\pm b$ is the hyperbola $x^2-y^2=a^2-b^2$.

12. Given the centre of a conic and two tangents; prove that the locus of the foci is a hyperbola.

[Take the two tangents as axes, their inclination being ω; let (x_1, y_1) and (x_2, y_2) be the foci, and (h, k) the given centre. Then $x_1+x_2=2h$ and $y_1+y_2=2k$; also, by Art. 270 (β), we have

$$y_1y_2\sin^2\omega=x_1x_2\sin^2\omega=(\text{semi-minor axis})^2.$$

From these equations, eliminating x_2 and y_2, we have

$$x_1^2-y_1^2=2hx_1-2ky_1.]$$

13. A given ellipse, of semi-axes a and b, slides between two perpendicular lines; prove that the locus of its focus is the curve

$$(x^2+y^2)(x^2y^2+b^4)=4a^2x^2y^2.$$

14. Conics are drawn touching both the axes, supposed oblique, at the same given distance a from the origin. Prove that the foci lie either on the straight line $x=y$, or on the circle

$$x^2+y^2+2xy\cos\omega=a(x+y).$$

15. Find the locus of the foci of conics which have a common point and a common director circle.

16. Find the locus of the focus of a rectangular hyperbola a diameter of which is given in magnitude and position.

17. Through a fixed point O chords POP' and QOQ' are drawn at right angles to one another to meet a given conic in P, P', Q, and Q'. Prove that $\frac{1}{PO \cdot OP'} + \frac{1}{QO \cdot OQ'}$ is constant.

18. A point is taken on the major axis of an ellipse whose abscissa is $ae \div \sqrt{2-e^2}$; prove that the sum of the squares of the reciprocals of the segments of any chord through it is constant.

19. Through a fixed point O is drawn a line OPP' to meet a conic in P and P'; prove that the locus of a point Q on OPP', such that $\frac{1}{OQ^2} = \frac{1}{OP^2} + \frac{1}{OP'^2}$ is another conic whose centre is O.

20. Prove Carnot's theorem, viz.: If a conic section cut the side BC of a triangle ABC in the points A' and A'', and, similarly, the side CA in B' and B'', and AB in C' and C'', then

$$BA' \cdot BA'' \cdot CB' \cdot CB'' \cdot AC' \cdot AC'' = CA' \cdot CA'' \cdot AB' \cdot AB'' \cdot BC' \cdot BC''.$$

[*Use Art.* 398.]

21. Obtain the equations giving the foci of the general conic by making use of the fact that, if S be a focus and PSP' any chord of the conic passing through it, then $\frac{1}{SP} + \frac{1}{SP'}$ is the same for all directions of the chord.

22. Obtain the equations for the foci also from the fact that the product of the perpendiculars drawn from them upon any tangent is the same for all tangents.

ANSWERS

6. $(-1, 5)$ and $(4, -3)$. 7. $(-\frac{4}{5}, -\frac{3}{5})$. 8. $\left(\frac{-59}{676}, \frac{66}{169}\right)$.

9. $(-4, -4)$ and $(-1, -1)$; $x+y+7=0$ and $x+y+3=0$.

15. If P be the given point, C the centre of the given director circle, and PCP' a diameter, the focus S is such that $PS \cdot P'S$ is constant.

16. If PP' be the given diameter and S a focus then $PS \cdot P'S$ is constant.

SOLUTIONS/HINTS

1. Let $\phi(x, y) \equiv ax^2 + by^2 - 1 = 0$ be the equation of the ellipse. Let A_1, A_2; B_1, B_2 be the points in which the tangents from (x', y') cut the axes. The points in which $y=0$ cuts $\phi(x, y) \times \phi(x', y') = u^2$ are given by

$$ax^2(by'^2-1)+2ax'x-1-\phi(x', y')=0.$$

$$\therefore\ OA_1 . OA_2=\frac{1+\phi(x', y')}{a(1-by'^2)}.$$

Similarly $$OB_1 . OB_2=\frac{1+\phi(x', y')}{b(1-ax'^2)}.$$

Also $OA_1 . OA_2 = OB_1 . OB_2$, since A_1, B_1, A_2, B_2 are concyclic.

Hence the required locus is $ab(x^2-y^2)+a-b=0$.

2. As in the last example the intercepts on the axis of x are given by

$$Ax^2(By'^2-1)+2Ax'x-1-\phi(x', y')=0;$$

$$\therefore\ k^2=\frac{(x_1-x_2)^2}{4}=\frac{Ax'^2+\{1+\phi(x', y')\}\{By'^2-1\}}{A(By'^2-1)^2}.$$

Therefore the required locus is

$$By^2(Ax^2+By^2-1)=Ak^2(By^2-1)^2.$$

3. The terms of the second degree in

$$\phi(x, y)\times\phi(x', y')=u^2,$$

are $$ax^2(\beta y'^2-1)-2a\beta x'y'xy+\beta y^2(ax'^2-1)+\ldots=0.$$

If the tangents are parallel to conjugate diameters of the second conic we have, by Art. 377,

$$ba(\beta y'^2-1)+2ha\beta x'y'+a\beta(ax'^2-1)=0.$$

Therefore the required locus is

$$ax^2+2hxy+by^2=\frac{a}{\alpha}+\frac{b}{\beta}.$$

4. The equation of all conics which touch the axes at the points where $ax+by=1$ cuts them is, by Art. 385 II,

$$(ax+by-1)^2-2\lambda xy=0.$$

The director circle of the conic given by the general equation of the second degree, when the axes are oblique, is

$$(x^2+y^2)(ab-h^2)+2x(bg-fh)+2y(af-gh)+c(a+b)-g^2-f^2$$
$$+2\cos\omega\left[(ab-h^2)xy+(af-gh)x+(bg-fh)y+gf-ch\right]=0.$$

[Arts. 390 and 93.]

Hence, for the above equation, it is

$$(x^2+y^2)(2ab-\lambda)-2bx-2ay + 2\cos\omega\{xy(2ab-\lambda)-ax-by+1\}=0,$$

i.e. $$(x^2+y^2+2xy\cos\omega)(2ab-\lambda) = 2(b+a\cos\omega)x+2y(a+b\cos\omega)-2\cos\omega,$$

which, for all values of λ, represents a system of circles having a common radical axis, viz.

$$x(b+a\cos\omega)+y(a+b\cos\omega)-\cos\omega=0.$$

5. Let (a, b) be the focus and $x\cos\alpha+y\sin\alpha=p$ the directrix. Then the equation of the parabola is

$$(x-a)^2+(y-b)^2=(x\cos\alpha+y\sin\alpha-p)^2.$$

Since it passes through $(0, 0)$, $(2h, 0)$, $(0, 2k)$,

$$\therefore\ a^2+b^2=p^2,\ (2h-a)^2+b^2=(2h\cos\alpha-p)^2,$$

and $$(2k-b)^2+a^2=(2k\cos\alpha-p)^2,$$

whence $$h\sin^2\alpha+p\cos\alpha=a, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and $$k\cos^2\alpha+p\sin\alpha=b. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Square and add;

$$\therefore\ h^2\sin^4\alpha+k^2\cos^4\alpha+2p(h\sin^2\alpha\cos\alpha+k\cos^2\alpha\sin\alpha)=0.$$

But $(p\cos\alpha, p\sin\alpha)$ is the point whose locus is required. Hence the required equation is

$$h^2y^4+k^2x^4+2xy(x^2+y^2)(hy+kx)=0.$$

The equation of the axis is $\cos\alpha(y-b)=\sin\alpha(x-a)$, since it goes through the focus (a, b) and is perpendicular to the directrix,

i.e. $$y=x\tan\alpha-\frac{a\sin\alpha-b\cos\alpha}{\cos\alpha}$$

$$=x\tan\alpha-\frac{h\sin^3\alpha-k\cos^3\alpha}{\cos\alpha} \text{ (from equations (i) and (ii))},$$

$$=x\tan\alpha-\frac{h\tan^3\alpha-k}{1+\tan^2\alpha}.$$

6. The equations for the foci are

$$\frac{(300x+160y-610)^2-(160x+144y-384)^2}{156}=\frac{(X)(Y)}{160}$$

$$=300x^2+320xy+144y^2-1220x-768y+199$$

$$=xX+yY-610x-384y+199.$$

The first pair give

$$40X^2-39XY-40Y^2=0, \text{ i.e. } (8X+5Y)(5X-8Y)=0.$$

$$\therefore\ 8X+5Y=0, \text{ whence } 8x+5y=17. \quad \ldots\ldots\text{(i)}$$

The second pair give, on reduction,

$$2xy-2x-3y+23=0. \quad \therefore\ 2x=\frac{3y-23}{y-1}.$$

Substitute in (i); $\therefore\ y^2-2y-15=0.$

$$\therefore\ y=5 \text{ or } -3, \text{ and } x=-1 \text{ or } 4.$$

The foci are therefore $(-1, 5)$, $(4, -3)$.

7. The equation for the focus is

$$\frac{(16x-12y+14)^2-(-12x+9y+7)^2}{7}=\frac{(X)(Y)}{-12}$$

$$=16x^2-24xy+9y^2+28x+14y+21.$$

The first pair gives $12Y^2-7XY-12X^2=0$,

i.e. $(4Y+3X)(3Y-4X)=0$; *i.e.* $70(3Y-4X)=0$.

Hence $\quad 3y-4x=\frac{7}{5}$ ....(i)

$$\therefore\ 16x-12y+14=\tfrac{42}{5}, \text{ and } -12x+9y+7=\tfrac{56}{5}.$$

Substitute in the second pair;

$$\therefore\ -\frac{42.56}{25.12}=xX+yY+14x+7y+21$$

$$=\tfrac{42}{5}x+\tfrac{56}{5}y+14x+7y+21,$$

or $\quad 80x+65y=-103.$ ....(ii)

From (i) and (ii), $\quad x=-\frac{4}{5},\ y=-\frac{3}{5}.$

8. The equations for the foci are

$$\frac{(144x-60y+\frac{67}{2})^2-(-60x+25y-21)^2}{119}=\frac{(X)(Y)}{-60}$$

$$=144x^2-120xy+25y^2+67x-42y+13.$$

The first pair give $60Y^2-119XY-60X^2=0$,

i.e. $(12Y+5X)(5Y-12X)=0$, so that $5Y-12X=0$.

Whence $12x-5y=-3$, and $X=-\frac{5}{2},\ Y=-6.$

Substitute in the second pair;

$$\therefore\ -\frac{5.6}{2.60}=xX+yY+\frac{67}{2}x-21y+13$$

$$=-\tfrac{5}{2}x-6y+\tfrac{67}{2}x-21y+13.$$

$$\therefore\ 124x-108y=-53.$$

Solving this with $12x-5y=-3$, we obtain

$$x=-\tfrac{59}{676},\ y=\tfrac{66}{169}.$$

9. The equations for the foci are

$$\frac{(x-3y-5)^2-(-3x+y-5)^2}{0}=\frac{(X)(Y)}{-3}$$

$$=xX+yY-5x-5y-19.$$

From the first pair, either $X=Y$ ....(i)

or $X=-Y.$ ....(ii)

If $X=Y$, then $x=y$, and

$$X=Y=-2x-5.$$

Substitute in the second pair;

$$\therefore\ \frac{(2x+5)^2}{-3}=-x(4x+10)-10x-19;$$

$\therefore\ x^2+5x+4=0$; $\therefore\ x=-4$ or -1, and $y=-4$ or -1.

The foci are therefore $(-1,-1)$ and $(-4,-4)$.

(ii) would give imaginary values.

The equation of the directrices which are the polars of the foci are

$$-3x-3y+5+5-19=0,\ \text{and}\ 3x+3y+20+20-19=0,$$

or $x+y+3=0$, and $x+y+7=0$.

10. The equations for the foci are

$$\frac{(ax+hy)^2-(hx+by)^2}{a-b}=\frac{(ax+hy)(hx+by)}{h}$$

$$=ax^2+2hxy+by^2-1,$$

which reduce to $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}=\dfrac{1}{h^2-ab}$.

11. The director-circle passes through the four points $(\pm a, \pm b)$; therefore the centre is $(0, 0)$.

Therefore if (x, y) be the coordinates of one focus, $(-x, -y)$ are those of the other.

Hence $(x-a)(-x-a) = \{\text{semi-minor axis}\}^2$

$$= (y-b)(-y-b). \quad [\text{Art. } 270\ (\beta).]$$

$$\therefore\ x^2 - y^2 = a^2 - b^2.$$

13. Let (x_1, y_1), (x_2, y_2) be the coordinates of the foci. Then

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (\text{distance between the foci})^2$$

$$= 4a^2e^2 = 4(a^2 - b^2). \quad \ldots\ldots\ldots\ldots(\text{i})$$

Also $\qquad x_1x_2 = b^2 = y_1y_2. \quad [\text{Art. } 270\ (\beta).] \ \ldots(\text{ii})$

Eliminating x_2 and y_2, we have

$$\left(x_1 - \frac{b^2}{x_1}\right)^2 + \left(y_1 - \frac{b^2}{y_1}\right)^2 = 4(a^2 - b^2).$$

$$\therefore\ (x_1^2 + y_1^2) + b^4\left(\frac{1}{x_1^2} + \frac{1}{y_1^2}\right) = 4a^2.$$

Therefore the required locus is $(x^2 + y^2)(x^2y^2 + b^4) = 4a^2x^2y^2$.

14. Let $x^2 + y^2 + 2\lambda xy - 2ax - 2ay + a^2 = 0$ be any one of the conics. The foci are given by

$$\phi(x, y) - (x + \lambda y - a)^2 = \phi(x, y) - (\lambda x + y - a)^2$$

$$= \frac{\lambda\phi(x, y) - (x + \lambda y - a)(\lambda x + y - a)}{\cos\omega}. \quad [\text{Arts. } 393 \text{ and } 175.]$$

From the first pair, $x = y$, or $1 + \lambda = \dfrac{2a}{x+y}$.

From the first and third equations,

$$\cos\omega\,\{2ay - y^2(1+\lambda)\} = xy(\lambda + 1) - ax - ay + a^2;$$

whence, on putting

$$1 + \lambda = \frac{2a}{x+y},\quad x^2 + y^2 + 2xy\cos\omega = a(x+y).$$

15. Let P be the common point, C the centre of the director circle whose radius is $\sqrt{\alpha^2+\beta^2}$, where α, β are the semi-axes of the conic. Then C and this radius are given. If S, H be the foci, then $SP.HP = CD^2 = \alpha^2+\beta^2 - CP^2 =$ given. Produce PC to P' so that $CP' = PC$. Then clearly $HP = SP'$, so that $SP.SP'$ is given $= \alpha^2+\beta^2-CP^2$.

Let $(a, 0)$ and $(-a, 0)$ be the coordinates of P and P', C being the origin. If (x, y) be the coordinates of S, then

$$\{(x+a)^2+y^2\}\{(x-a)^2+y^2\} = \{r^2-a^2\}^2,$$

where r is the radius of the director circle.

16. As in the last example, $SP.SP' = SP.HP = CP^2$, in a rectangular hyperbola.

Hence the locus is $\{(x+a)^2+y^2\}\{(x-a)^2+y^2\} = a^4$.

17. See the example of Art. 401;

$$\frac{1}{PO.OP'} = \frac{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}{c},$$

and, on changing θ into $\theta+\frac{\pi}{2}$,

$$\frac{1}{QO.OQ'} = \frac{a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta}{c};$$

$$\therefore \frac{1}{PO.OP'} + \frac{1}{QO.OQ'} = \frac{a+b}{c}.$$

18. Remove the origin to the given point, and change to polars.

The equation then is

$$r^2\left\{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right\} + \frac{2r.e\cos\theta}{a\sqrt{2-e^2}} + \frac{2(e^2-1)}{2-e^2} = 0.$$

$$\therefore \frac{1}{OP}+\frac{1}{OP'} = -\frac{e\cos\theta\sqrt{2-e^2}}{a(e^2-1)},$$

and $$\frac{1}{OP.OP'} = \frac{(2-e^2)(b^2\cos^2\theta+a^2\sin^2\theta)}{2a^2b^2(e^2-1)}.$$

$$\therefore \frac{1}{OP^2}+\frac{1}{OP'^2}=\frac{e^2\cos^2\theta\,(2-e^2)}{a^2(e^2-1)^2}-\frac{(2-e^2)(b^2\cos^2\theta+a^2\sin^2\theta)}{a^2b^2(e^2-1)}$$

$$=\frac{2-e^2}{a^2(e^2-1)^2}\left[e^2\cos^2\theta-\frac{b^2\cos^2\theta+a^2\sin^2\theta}{b^2}(e^2-1)\right]$$

$$=\frac{(2-e^2)}{a^2(e^2-1)^2}=\frac{a^2+b^2}{b^4}.$$

19. As in the example of Art. 401,

$$\frac{1}{OP}+\frac{1}{OP'}=\frac{-2(g\cos\theta+f\sin\theta)}{c},$$

and $$\frac{1}{OP\,.\,OP}=\frac{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}{c}.$$

$$\therefore \frac{1}{r^2}=\frac{1}{OQ^2}=\frac{1}{OP^2}+\frac{1}{OP'^2}$$

$$=\frac{4(g\cos\theta+f\sin\theta)^2}{c^2}-\frac{2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)}{c},$$

which is of the form $Ax^2+2Hxy+By^2=C$.

20. Let r_1, r_2, r_3 be the semi-diameters which are parallel to BC, CA, AB. Then by Art. 398,

$$\frac{BA'\,.\,BA''}{BC'\,.\,BC''}=\frac{r_1^2}{r_3^2},\ \text{etc.}$$

The result follows by multiplication.

21. Taking equation (2) of Art. 397, if S be (x', y'), then

$$\left\{\frac{1}{SP}+\frac{1}{SP'}\right\}^2=\left\{\frac{1}{r_1}-\frac{1}{r_2}\right\}^2=\left\{\frac{1}{r_1}+\frac{1}{r_2}\right\}^2-\frac{4}{r_1r_2}$$

$$=4\left[\left\{\frac{(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta}{\phi(x', y')}\right\}^2-\frac{a\cos^2\theta+2h\cos\theta\sin\theta+b\cos^2\theta}{\phi(x', y')}\right].$$

If this is invariable for all values of θ, equate the values when $\theta=0,\ \frac{\pi}{4},\ \frac{\pi}{2}$.

$$\therefore \frac{(ax+hy+g)^2}{\phi(x,y)} - a = \frac{(hx+by+f)^2}{\phi(x,y)} - b$$
$$= \frac{1}{2}\left[\frac{\{(ax+hy+g)+(hx+by+f)\}^2}{\phi(x,y)} - (a+b+2h)\right].$$

These equations are

$$\frac{X^2}{\phi} - a = \frac{Y^2}{\phi} - b = \frac{(X+Y)^2}{2\phi} - \frac{a+b+2h}{2}.$$

The first pair gives $\dfrac{X^2 - Y^2}{\phi} = a - b.$

The first two added, and equated to twice the third, give

$$\frac{X^2+Y^2}{\phi} - a - b = \frac{(X+Y)^2}{\phi} - (a+b+2h). \quad \therefore\ h = \frac{XY}{\phi}.$$

Hence the equations give $\dfrac{X^2 - Y^2}{a-b} = \dfrac{XY}{h} = \phi.$

These are the same as the equations of Art. 393.

22. The line $x\cos\theta + y\sin\theta = p$ will touch the conic given by the general equation if

$$Cp^2 - 2p(G\cos\theta + F\sin\theta)$$
$$+ A\cos^2\theta + 2H\cos\theta\sin\theta + B\sin^2\theta = 0 \ldots \text{(i)}$$

where A, B, C, etc. are the minors of a, b, c, etc. in

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \quad \text{[Art. 374].}$$

Hence the product of the perpendiculars from (x, y) upon the parallel tangents

$$x\cos\theta + y\sin\theta = p_1, \text{ and } x\cos\theta + y\sin\theta = p_2$$

is $(x\cos\theta + y\sin\theta)^2 - (p_1 + p_2)(x\cos\theta + y\sin\theta) + p_1 p_2$,

which

$$\propto C(x\cos\theta + y\sin\theta)^2 - 2(G\cos\theta + F\sin\theta)(x\cos\theta + y\sin\theta)$$
$$+ A\cos^2\theta + B\sin^2\theta + 2H\cos\theta\sin\theta, \text{ by (i).}$$

Put $\theta = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}$ and equate;

$$\therefore\ Cx^2 - 2xG + A = Cy^2 - 2yF + B$$
$$= \tfrac{1}{2}\left[C(x+y)^2 - 2(x+y)(G+F) + A + B + 2H\right],$$

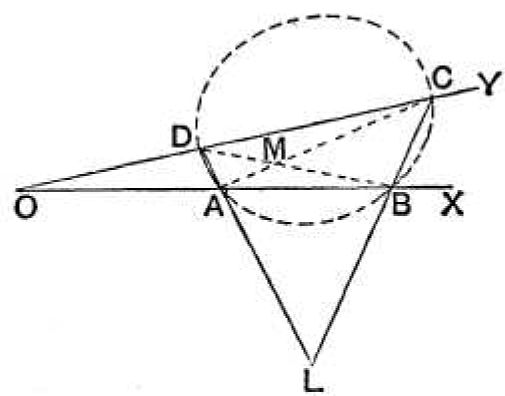
or
$$C(x^2 - y^2) - 2Gx + 2Fy + A - B = 0,$$

and
$$Cxy - Fx - Gy + H = 0.$$

The equations of Art. 393 reduce to these, on multiplying out and collecting the terms.

405. *General equation to conics passing through four given points.*

Let A, B, C, and D be the four points, and let BA and CD meet in O. Take OAB and ODC as the axes, and let $OA = \lambda$, $OB = \lambda'$, $OD = \mu$, and $OC = \mu'$.



Let any conic passing through the four points be

$$ax^2 + 2h'xy + by^2 + 2gx + 2fy + c = 0 \ldots (1).$$

If we put $y = 0$ in this equation the roots of the resulting equation must be λ and λ'.

Hence $2g = -a(\lambda + \lambda')$ and $c = a\lambda\lambda'$,

i.e. $a = \frac{c}{\lambda\lambda'}$, and $2g = -c\frac{\lambda + \lambda'}{\lambda\lambda'}$.

Similarly $b = \frac{c}{\mu\mu'}$, and $2f = -c\frac{\mu + \mu'}{\mu\mu'}$.

On substituting in (1) we have

$$\mu\mu' x^2 + 2hxy + \lambda\lambda' y^2 - \mu\mu'(\lambda + \lambda')x$$
$$- \lambda\lambda'(\mu + \mu')y + \lambda\lambda'\mu\mu' = 0 \ldots\ldots(1),$$

where $h = h'\frac{\lambda\lambda'\mu\mu'}{c}$.

This is the required equation, h being a constant as yet undetermined and depending on which of the conics through A, B, C, and D we are considering.

406. Aliter. We have proved in Art. 383 that the equation $kLN = MR$, k being any constant, represents any conic circumscribing the quadrilateral formed by the four straight lines $L = 0$, $M = 0$, $N = 0$, and $R = 0$ taken in this order.

With the notation of the previous article the equations to the four lines AB, BC, CD, and DA are

$$y = 0, \quad \frac{x}{\lambda'} + \frac{y}{\mu'} - 1 = 0, \quad x = 0,$$

and

$$\frac{x}{\lambda} + \frac{y}{\mu} - 1 = 0.$$

The equation to any conic circumscribing the quadrilateral $ABCD$ is therefore

$$kxy = \left(\frac{x}{\lambda'} + \frac{y}{\mu'} - 1\right)\left(\frac{x}{\lambda} + \frac{y}{\mu} - 1\right) \ldots\ldots\ldots\ldots(1),$$

i.e.

$$\mu\mu' x^2 + xy(\lambda\mu' + \lambda'\mu - k\lambda\lambda'\mu\mu') + \lambda\lambda' y^2$$
$$- \mu\mu'(\lambda + \lambda')x - \lambda\lambda'(\mu + \mu')y + \lambda\lambda'\mu\mu' = 0.$$

On putting $\lambda\mu' + \lambda'\mu - k\lambda\lambda'\mu\mu'$ equal to another constant $2h$ we have the equation (1) of the previous article.

407. *Only one conic can be drawn through any five points.*

For the general equation to a conic through four points is (1) of Art. 405.

If we wish it to pass through a fifth point, we substitute the coordinates of this fifth point in this equation, and thus obtain the corresponding value of h. Except when three of the five points lie on a straight line a value of h will always be found, and only one.

Ex. *Find the equation to the conic section which passes through the five points A, B, C, D, and E, whose coordinates are* (1, 2), (3, −4), (−1, 3), (−2, −3), *and* (5, 6).

The equations to AB, BC, CD, and DA are easily found to be

$$y+3x-5=0,\ \ 4y+7x-5=0,\ \ 6x-y+9=0,\ \text{ and }\ 5x-3y+1=0.$$

The equation to any conic through the four points A, B, C, and D is therefore

$$(y+3x-5)(6x-y+9)=\lambda(4y+7x-5)(5x-3y+1)\ldots\ldots(1).$$

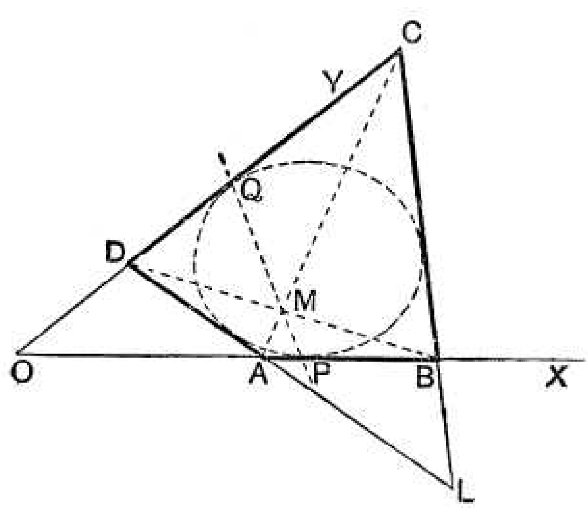
If this conic pass through the point E, the equation (1) must be satisfied by the values $x=5$ and $y=6$.

We thus have $\lambda=\frac{11}{9}$ and, on substitution in (1), the required equation is

$$223x^2-38xy-123y^2-171x+83y+350=0,$$

which represents a hyperbola.

408. *To find the general equation to a conic section which touches four given straight lines, i.e. which is inscribed in a given quadrilateral.*



Let the four straight lines form the sides of the quadrilateral $ABCD$. Let BA and CD meet in O, and take OAB and ODC as the axes of x and y, and let the equations to the other two sides BC and DA be

$$l_1x + m_1y - 1 = 0, \text{ and } l_2x + m_2y - 1 = 0.$$

Let the equation to the straight line joining the points of contact of any conic touching the axes at P and Q be

$$ax + by - 1 = 0.$$

By Art. 385, II, the equation to the conic is then

$$2\lambda xy = (ax + by - 1)^2 \text{..................(1).}$$

The condition that the straight line BC should touch this conic is, as in Art. 374, found to be

$$\lambda = 2(a - l_1)(b - m_1) \text{..................(2).}$$

Similarly, it will be touched by AD if

$$\lambda = 2(a - l_2)(b - m_2) \text{..................(3).}$$

The required conic has therefore (1) as its equation, the values of a and b being given in terms of the quantity λ by means of (2) and (3).

Also λ is any quantity we may choose. Hence we have the system of conics touching the four given lines.

If we solve (2) and (3), we obtain

$$\frac{2b - (m_1 + m_2)}{m_1 - m_2} = -\frac{2a - (l_1 + l_2)}{l_1 - l_2} = \pm\sqrt{1 - \frac{2\lambda}{(l_1 - l_2)(m_1 - m_2)}}.$$

409. *The conic* $LM = R^2$, *where* $L = 0$, $M = 0$, *and* $R = 0$ *are the equations of straight lines.*

The equation $LM = 0$ represents a conic, viz. two straight lines.

Hence, by Art. 385, II, the equation

$$LM = R^2 \text{........................(1),}$$

represents a conic touching the straight lines $L = 0$, and $M = 0$, where $R = 0$ meets them.

Thus $L=0$ and $M=0$ are a pair of tangents and $R=0$ the corresponding chord of contact.

Every point which satisfies the equations $M=\mu^2L$ and $R=\mu L$ clearly lies on (1).

Hence the point of intersection of the straight lines $M=\mu^2L$ and $R=\mu L$ lies on the conic (1) for all values of μ. This point may be called the point "μ."

410. *To find the equation to the straight line joining two points "μ" and "μ'" and the equation to the tangent at the point "μ."*

Consider the equation

$$aL+bM+R=0 \qquad \text{.................. (1).}$$

Since it is of the first degree and contains two constants a and b, at our disposal, it can be made to represent any straight line.

If it pass through the point "μ" it must be satisfied by the substitutions $M=\mu^2L$ and $R=\mu L$.

Hence $$a+b\mu^2+\mu=0 \qquad \text{.................... (2).}$$

Similarly, if it pass through the point "μ'" we have

$$a+b\mu'^2+\mu'=0 \qquad \text{.................... (3).}$$

Solving (2) and (3), we have

$$\frac{a}{\mu\mu'}=b=\frac{-1}{\mu+\mu'}.$$

On substitution in (1), the equation to the joining line is

$$L\mu\mu'+M-(\mu+\mu')R=0.$$

By putting $\mu'=\mu$ we have, as the equation to the tangent at the point "μ,"

$$L\mu^2+M-2\mu R=0.$$

EXAMPLES XLV

1. Prove that the locus of the foot of the perpendicular let fall from the origin upon tangents to the conic $ax^2+2hxy+by^2=2x$ is the curve $(h^2-ab)(x^2+y^2)^2+2(x^2+y^2)(bx-hy)+y^2=0.$

2. In the conic $ax^2+2hxy+by^2=2y$, prove that the rectangle contained by the focal distances of the origin is $\frac{1}{ab-h^2}$.

3. Tangents are drawn to the conic $ax^2+2hxy+by^2=2x$ from two points on the axis of x equidistant from the origin; prove that their four points of intersection lie on the conic $by^2+hxy=x$.

If the tangents be drawn from two points on the axis of y equidistant from the origin, prove that the points of intersection are on a straight line.

4. A system of conics is drawn to pass through four fixed points; prove that

(1) the polars of a given point all pass through a fixed point,

and (2) the locus of the pole of a given line is a conic section.

5. Find the equation to the conic passing through the origin and the points (1, 1), (−1, 1), (2, 0), and (3, −2). Determine its species.

6. Prove that the locus of the centre of all conics circumscribing the quadrilateral formed by the straight lines $y=0$, $x=0$, $x+y=1$, and $y-x=2$ is the conic $2x^2-2y^2+4xy+5y-2=0$.

7. Prove that the locus of the centres of all conics, which pass through the centres of the inscribed and escribed circles of a triangle, is the circumscribing circle of the triangle.

8. Prove that the locus of the extremities of the principal axes of all conics, which can be described through the four points $(\pm a, 0)$ and $(0, \pm b)$, is the curve

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)(x^2+y^2)=x^2-y^2.$$

9. A, B, C, and D are four fixed points and AB and CD meet in O; any straight line passing through O meets AD and BC in R and R' respectively, and any conic passing through the four given points in S and S'; prove that

$$\frac{1}{OR}+\frac{1}{OR'}=\frac{1}{OS}+\frac{1}{OS'}.$$

10. Prove that, in general, two parabolas can be drawn through four points, and that either two, or none, can be drawn.

[For a parabola we have $h=\pm\sqrt{\lambda\lambda'\mu\mu'}$.]

11. Prove that the locus of the centres of the conics circumscribing a quadrilateral *ABCD* (Fig. Art. 405) is a conic passing through the vertices *O*, *L*, and *M* of the quadrilateral and through the middle points of *AB*, *AC*, *AD*, *BC*, *BD*, and *CD*.

Prove also that its asymptotes are parallel to the axes of the parabolas through the four points.

[The required locus is obtained by eliminating h from the equations $2\mu\mu'x+2hy-\mu\mu'(\lambda+\lambda')=0$, and $2hx+2\lambda\lambda'y-\lambda\lambda'(\mu+\mu')=0$.]

12. By taking the case when $\lambda\lambda'=-\mu\mu'$ and when *AB* and *CD* are perpendicular (in which case *ABC* is a triangle having *D* as its orthocentre and *AL*, *BM*, and *CO* are the perpendiculars on its sides), prove that all conics passing through the vertices of a triangle and its orthocentre are rectangular hyperbolas.

From Ex. 11 prove also that the locus of its centre is the nine point circle of the triangle.

13. Prove that the triangle *OML* (Fig. Art. 405) is such that each angular point is the pole of the opposite side with respect to any conic passing through the angular points *A*, *B*, *C*, and *D* of the quadrilateral.

[Such a triangle is called a **Self Conjugate Triangle.**]

14. Prove that only one rectangular hyperbola can be drawn through four given points. Prove also that the nine point circles of the four triangles that can be formed by four given points meet in a point, viz., the centre of the rectangular hyperbola passing through the four points.

15. By using the result of Art. 374, prove that in general, two conics can be drawn through four points to touch a given straight line.

A system of conics is inscribed in the same quadrilateral; prove that

16. the locus of the pole of a given straight line with respect to this system is a straight line.

17. the locus of their centres is a straight line passing through the middle points of the diagonals of the quadrilateral.

18. Prove that the triangle formed by the three diagonals *OL*, *AC*, and *BD* (Fig. Art. 408) is such that each of its angular points is the pole of the opposite side with respect to any conic inscribed in the quadrilateral.

19. Prove that only one parabola can be drawn to touch any four given lines.

Hence prove that, if the four triangles that can be made by four lines be drawn, the orthocentres of these four straight lines lie on a straight line, and their circumcircles meet in a point.

ANSWERS

5. $6x^2+12xy+7y^2-12x-13y=0.$

17. The narrow ellipse (Art. 408), which is very nearly coincident with the straight line BD, is one of the conics inscribed in the quadrilateral, and its centre is the middle point of BD. This middle point, and similarly the middle points of AC and OL, therefore lie on the centre-locus.

SOLUTIONS/HINTS

1. The line $x\cos\theta + y\sin\theta = r$ will touch the given conic if the equation

$$ax^2 + 2hx \cdot \frac{r - x\cos\theta}{\sin\theta} + b\left(\frac{r - x\cos\theta}{\sin\theta}\right)^2 = 2x$$

has equal roots, the condition for which is

$$\{hr\sin\theta - br\cos\theta - \sin^2\theta\}^2 = br^2(a\sin^2\theta - 2h\sin\theta \cdot \cos\theta + b\cos^2\theta),$$

which is the polar equation of the locus required.

Changing to Cartesians, we have

$$(hy-bx)^2 + \frac{y^4}{(x^2+y^2)^2} + \frac{2y^2}{x^2+y^2}(bx-hy) = b(ay^2 - 2hxy + bx^2),$$

or $\qquad (h^2-ab)(x^2+y^2)^2 + y^2 + 2(x^2+y^2)(bx-hy) = 0.$

2. The ordinate of the centre is

$$\frac{a}{ab-h^2}. \quad \text{[Art. 352.]}$$

Therefore the diameter, DCD', parallel to the tangent $y=0$, cuts the curve in points given by

$$ax^2 + \frac{2ah}{ab-h^2} \cdot x + \frac{a^2b}{(ab-h^2)^2} - \frac{2a}{ab-h^2} = 0.$$

$$\therefore\ CD^2 = \left(\frac{x_1-x_2}{2}\right)^2 = \frac{h^2}{(ab-h^2)^2} - \frac{ab}{(ab-h^2)^2} + \frac{2}{(ab-h^2)} = \frac{1}{ab-h^2}.$$

$$SO \cdot S'O = CD^2 = \frac{1}{ab-h^2} \quad \text{[Art. 287].}$$

3. As in Art. 374, the condition that the line $y = m(x - c)$ should touch the conic is

$$m^2c^2(h^2 - ab) + 2mc(h + bm) + 1 = 0.$$

Hence the equation of the tangents through the point $(c, 0)$ is

$$c^2y^2(h^2 - ab) + 2chy(x - c) + 2cby^2 + (x - c)^2 = 0,$$

and similarly those through $(-c, 0)$ are

$$c^2y^2(h^2 - ab) - 2chy(x + c) - 2cby^2 + (x + c)^2 = 0.$$

On subtraction, the four points of intersection lie on

$$by^2 + hxy = x.$$

From the condition of Art. 374, the line $x = m(y - c)$ will touch the conic if $c^2m(h^2 - ab) + m - 2chm - 2cb = 0$.

Hence the other tangent through the point $(0, c)$ is

$$(h^2 - ab)c^2x + x - 2cb(y - c) - 2chx = 0,$$

and that through $(-c, 0)$ is

$$(h^2 - ab)c^2x + x + 2cb(y + c) + 2chx = 0.$$

Whence, on subtraction, the point of intersection lies on

$$by + hx = 0.$$

4. (1) Let the fixed point be the origin, and $S - \lambda S' = 0$ the equation of the system of conics. [Art. 380.]

The equation of its polar is

$$gx + fy + c - \lambda(g'x + f'y + c') = 0,$$

which passes through the intersection of the lines

$$gx + fy + c = 0 \text{ and } g'x + f'y + c' = 0.$$

(2) Let $y = 0$ be the given line.

It will be identical with the polar of (x', y') if, (Art. 375),

$$ax' + hy' + g - \lambda(a'x' + b'y' + g') = 0, \text{ and}$$

$$gx' + fy' + c - \lambda(g'x' + f'y' + c') = 0.$$

Eliminating λ, the locus of (x', y') is a conic.

5. The equations of the sides of the quadrilateral are easily found to be

$$y - 1 = 0;\ x + y - 2 = 0;\ y + 2x - 4 = 0;\ \text{and } 3x + 4y = 1.$$

Therefore the equation of any conic passing through the four points is $(y-1)(y+2x-4)=\lambda(x+y-2)(3x+4y-1)$, which passes through (0, 0) if $\lambda=2$.

Hence etc. The conic is an ellipse, since $h^2 < ab$.

6. The centre of the conic $x(y-x-2)=\lambda y(x+y-1)$ is given by (Art. 352)

$$\lambda x - x + 2\lambda y - \lambda = 0, \text{ and } 2x + \lambda y - y + 2 = 0.$$

Eliminating λ, we obtain, for the locus of the centre,

$$2x^2 + 4xy - 2y^2 + 5y - 2 = 0.$$

7. The centre of the conic of Art. 405 is given by

$$2\mu\mu' x + 2hy - \mu\mu'(\lambda+\lambda') = 0,$$

and $$2hx + 2\lambda\lambda' y - \lambda\lambda'(\mu+\mu') = 0.$$

Eliminating h, we have

$$2(\mu\mu' x^2 - \lambda\lambda' y^2) - \mu\mu'(\lambda+\lambda')x + \lambda\lambda'(\mu+\mu')y = 0.$$

If the axes AB and CD are at right angles and AD is perpendicular to BC, then $\mu\mu' + \lambda\lambda' = 0$, and the centre-locus becomes

$$2(x^2+y^2) - (\lambda+\lambda')x - (\mu+\mu')y = 0,$$

which is the equation to the nine-point circle of the triangle ABC, since it passes through the middle points of AB, BC and CA, whose coordinates are

$$\left(\frac{\lambda+\lambda'}{2},\ 0\right),\ \left(\frac{\lambda'}{2},\ \frac{\mu}{2}\right) \text{ and } \left(\frac{\lambda}{2},\ \frac{\mu'}{2}\right).$$

This proves the theorem. For, if A, B, C are the e-centres of a triangle, the latter is the pedal triangle of ABC, and hence its circumcircle is the nine-point circle of ABC.

8. All the conics through the four points $(\pm a,\ 0)$, $(0,\ \pm b)$ are of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\lambda xy = 1.$$

If an axis is inclined at α, the line $x\cos\alpha + y\sin\alpha = p$ touches the conic at the point $(p\cos\alpha,\ p\sin\alpha)$, and is therefore identical with

$$x\left(\frac{p\cos\alpha}{a^2} + \lambda p\sin\alpha\right) + y\left(\lambda p\cos\alpha + \frac{p\sin\alpha}{b^2}\right) = 1.$$

$$\therefore\ \frac{1}{a^2} + \lambda\tan\alpha = \frac{1}{b^2} + \lambda\cot\alpha = \frac{1}{p^2}.$$

Eliminating λ, the polar equation of the required locus is

$$\left(\frac{1}{r^2}-\frac{1}{a^2}\right)\cot\theta=\left(\frac{1}{r^2}-\frac{1}{b^2}\right)\tan\theta,$$

or, in Cartesians,

$$\left\{\frac{1}{x^2+y^2}-\frac{1}{a^2}\right\}\frac{x}{y}=\left\{\frac{1}{x^2+y^2}-\frac{1}{b^2}\right\}\frac{y}{x},$$

i.e.

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)(x^2+y^2)=x^2-y^2.$$

9. Let $\left(\frac{x}{a}+\frac{y}{b}-1\right)\left(\frac{x}{a'}+\frac{y}{b'}-1\right)+\lambda xy=0$

be the equation of any conic through the four points. [Art. 406.]

Changing to polar coordinates, this equation is

$$r^2(\quad)-r\left\{\cos\theta\left(\frac{1}{a}+\frac{1}{a'}\right)+\sin\theta\left(\frac{1}{b}+\frac{1}{b'}\right)\right\}+1=0.$$

$$\therefore\ \frac{1}{OS}+\frac{1}{OS'}=\cos\theta\left(\frac{1}{a}+\frac{1}{a'}\right)+\sin\theta\left(\frac{1}{b}+\frac{1}{b'}\right).$$

Also $\frac{1}{OR}=\frac{\cos\theta}{a}+\frac{\sin\theta}{b}$, and $\frac{1}{OR'}=\frac{\cos\theta}{a'}+\frac{\sin\theta}{b'}$.

[Art. 105.]

$$\therefore\ \frac{1}{OR}+\frac{1}{OR'}=\frac{1}{OS}+\frac{1}{OS'}.$$

10. There will be two values of h or none according as $\lambda\lambda'\mu\mu'$ is positive or negative.

11. As in Ex. 7, the centre locus is

$$2(\mu\mu' x^2-\lambda\lambda' y^2)-\mu\mu'(\lambda+\lambda')x-\lambda\lambda'(\mu+\mu')y=0.$$

The locus passes through the point $\left(\frac{\lambda+\lambda'}{2},\ 0\right)$, *i.e.* through the middle point of the line joining two of the fixed points, and therefore through that of the line joining any other two. It also passes through the origin and hence by symmetry through L and M.

Its asymptotes and the axes of the parabolas through the four points are parallel to $\sqrt{\mu\mu'}x\pm\sqrt{\lambda\lambda'}y=0$. (Cf. Ex. 10.)

12. See Ex. 7.

13. The polar of (0, 0) is

$$x\left(\frac{1}{\lambda}+\frac{1}{\lambda'}\right)+y\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=2,$$

which passes through the intersections of

$$\frac{x}{\lambda}+\frac{y}{\mu}=1 \text{ and } \frac{x}{\lambda'}+\frac{y}{\mu'}=1,$$

and of

$$\frac{x}{\lambda'}+\frac{y}{\mu}=1 \text{ and } \frac{x}{\lambda}+\frac{y}{\mu'}=1.$$

Hence etc. by symmetry.

14. Let $S=0$ and $S'=0$ be any two of the conics. Then $S-\lambda S'=0$ for different values of λ represents all the conics through their intersections. It will be a rectangular hyperbola if $a+b-\lambda(a'+b')=0$, and hence there is only one value of λ.

By Ex. 12, each of the four nine-point circles passes through the centre.

15. Apply the condition of Art. 374 to the conic of Art. 405; since it contains h^2, there will in general be two conics.

16. If $lx+my=1$ be the polar of (x', y') with regard to the conic of Art. 408, then

$$\frac{a^2x'+(ab-\lambda)y'-a}{l}=\frac{(ab-\lambda)x'+b^2y'-b}{m}=ax'+by'-1.$$

$$\therefore (a-l)(ax'+by'-1)=\lambda y' \text{ and } (b-m)(ax'+by'-1)=\lambda x'.$$

$$\therefore \frac{x'}{b-m}=\frac{y'}{a-l}=\frac{ax'+by'-1}{\lambda}$$

$$=\frac{ax'+by'-1}{2(a-l_1)(b-m_1)}, \text{ by Art. 408 (2)}, =\frac{1}{\rho} \text{ (say)}. \ldots(1)$$

$$\therefore b=\rho x'+m, \text{ and } a=\rho y'+l.$$

Substitute for a and b in the last pair of equations (1), we have

$$\rho\{x'(\rho y'+l)+y'(\rho x'+m)-1\}=2(\rho y'+l-l_1)(\rho x'+m-m_1),$$

i.e. $\rho\{(2l_1-l)x'+(2m_1-m)y'-1\}=2(l-l_1)(m-m_1).$

Similarly,

$$\rho\{(2l_2-l)x'+(2m_2-m)y'-1\}=2(l-l_2)(m-m_2).$$

Eliminating ρ, the locus of (x', y') is a straight line.

17. The centre of the conic of Art 408 is given by

$$a(ax+by-1)=\lambda y, \text{ and } b(ax+by-1)=\lambda x,$$

whence $ax=by$.

$$\therefore a(2ax-1)=\lambda y=2y(a-l_1)(b-m_1)=2(a-l_1)(ax-m_1y).$$

$$\therefore a(2l_1x+2m_1y-1)=2l_1m_1y.$$

Similarly, $\quad a(2l_2x+2m_2y-1)=2l_2m_2y.$

$$\therefore 2x\left(\frac{1}{m_1}-\frac{1}{m_2}\right)+2y\left(\frac{1}{l_1}-\frac{1}{l_2}\right)=\frac{1}{l_1m_1}-\frac{1}{l_2m_2},$$

which passes through $\left(\frac{1}{2l_1}, \frac{1}{2m_2}\right)$; $\left(\frac{1}{2l_2}, \frac{1}{2m_1}\right)$, *i.e.* through the middle points of BD and AC.

18. The equation to OL is $(l_1-l_2)x+(m_1-m_2)y=0$ and, if it be identical with the polar of (x', y') with regard to the conic of Art. 408, then

$$\frac{a^2x'+(ab-\lambda)y'-a}{l_1-l_2}=\frac{(ab-\lambda)x'+b^2y'-b}{m_1-m_2},$$

and $ax'+by'-1=0$, whence $\dfrac{y'}{l_1-l_2}=\dfrac{x'}{m_1-m_2}$.

Also, by subtracting equations (2) and (3) of Art. 408,

$$a(m_1-m_2)+b(l_1-l_2)=l_1m_1-l_2m_2.$$

$$\therefore \frac{x'}{m_1-m_2}=\frac{y'}{l_1-l_2}=\frac{ax'+by'}{a(m_1-m_2)+b(l_1-l_2)}=\frac{1}{l_1m_1-l_2m_2}.$$

But these are the coordinates of the point M, since the latter point is the intersection of the two straight lines

$$l_1x+m_2y=1 \text{ and } l_2x+m_1y=1.$$

Hence M is the pole of OL, etc.

19. The conic of Art. 408 is a parabola only if $\lambda=2ab$. The orthocentres lie on its directrix and the circumcircles all pass through its focus. [Arts. 232, 234.]