Exercise 7.4

Answer 1E.

(a)

Consider the fraction $\dfrac{1+6x}{(4x-3)(2x+5)}$

The denominator $Q(x)$ is a product of distinct linear factors

This means that

$$Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

Where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots A_k$ such that

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k} \qquad \ldots\ldots(1)$$

Here, $(4x-3)$ and $(2x+5)$ are two distinct linear factors

So we use two constants A and B

Therefore, take partial fractions of the fraction $\dfrac{1+6x}{(4x-3)(2x+5)}$

$$\frac{1+6x}{(4x-3)(2x+5)}=\frac{A}{4x-3}+\frac{B}{2x+5}$$

(b)

Consider the fraction $\dfrac{10}{5x^2-2x^3}$

It can be written as by taking x^2 common in the denominator

$$\frac{10}{5x^2-2x^3}=\frac{10}{x^2(5-2x)}$$

The denominator $Q(x)$ is a product of linear factors, some of which are repeated:

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ in equation (1), we would use

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

Here the fraction $\dfrac{10}{5x^2-2x^3}=\dfrac{10}{x^2(5-2x)}$

In this fraction, there are three linear factors with repeated factor x use three constants here $A, B,$ and C

So, use the partial fraction method, write the fraction $\dfrac{10}{x^2(5-2x)}$ as

$$\frac{10}{x^2(5-2x)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{5-2x}$$

Answer 2E.

Write the partial fraction decomposition of the following functions without determining the numerical values of the coefficients.

a) Consider the expression, $\dfrac{x}{x^2+x-2}$.

The denominator can be written as follows:

$$x^2+x-2=(x+2)(x-1)$$

Write the partial fraction decomposition of the function.

$$\frac{x}{x^2+x-2}=\frac{x}{(x+2)(x-1)}$$
$$=\boxed{\frac{A}{(x+2)}+\frac{B}{(x-1)}}$$

b) Consider the expression, $\dfrac{x^2}{x^2+x+2}$.

Since the degree of the numerator is equal to the degree of the denominator.

So, first perform the long division.

$$\begin{array}{r} 1 \\ x^2+x+2\overline{)\,x^2\qquad} \\ \underline{x^2+x+2} \\ -x-2 \end{array}$$

Therefore, the given expression can be written as follows:

$$\frac{x^2}{x^2+x+2}=1+\frac{-x-2}{x^2+x+2}$$

Now, decompose the second term (the denominator cannot be factored however) and find the partial fraction decomposition of the function.

$$\begin{aligned}\frac{x^2}{x^2+x+2}&=1+\frac{-x-2}{x^2+x+2}\\ &=\boxed{1+\frac{Ax+B}{x^2+x+2}}\end{aligned}$$

Answer 3E.

Write the partial fraction decomposition of the following functions without determining the numerical values of the coefficients.

a) Consider the expression, $\dfrac{x^4+1}{x^5+4x^3}$

The denominator can be written as follows:

$$x^5+4x^3=x^3\left(x^2+4\right)$$

Therefore, write the partial fraction decomposition.

$$\begin{aligned}\frac{x^4+1}{x^5+4x^3}&=\frac{x^4+1}{x^3\left(x^2+4\right)}\\ &=\boxed{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+4}}\end{aligned}$$

b) Consider the expression, $\dfrac{1}{\left(x^2-9\right)^2}$.

The denominator can be written as follows:

$$\begin{aligned}\left(x^2-9\right)^2&=\left[(x+3)(x-3)\right]^2\\ &=(x+3)^2(x-3)^2\end{aligned}$$

Therefore, write the partial fraction decomposition.

$$\begin{aligned}\frac{1}{\left(x^2-9\right)^2}&=\frac{1}{(x+3)^2(x-3)^2}\\ &=\boxed{\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{x-3}+\frac{D}{(x-3)^2}}\end{aligned}$$

Answer 4E.

(a) Given $\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}$

$$= x^2+\frac{2x-1}{x^2-2x+1}$$

$$= x^2+\frac{2x-1}{(x-1)^2}$$

$$= x^2+\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

(b) Given $\frac{x^2-1}{x^3+x^2+x}=\frac{x^2-1}{x(x^2+x+1)}$

$$=\frac{A}{x}+\frac{Bx+C}{x^2+x+1}$$

Answer 5E.

(a)

Consider the expression, $\frac{x^6}{x^2-4}$.

The objective is to write partial decomposition of $\frac{x^6}{x^2-4}$.

The degree of the numerator is greater than the degree of the denominator. So divide the numerator by denominator.

$$\begin{array}{r|l} & x^4+4x^2+16 \\ x^2-4 & x^6 \\ & x^6-4x^4 \\ \hline & 4x^4 \\ & 4x^4-16x^2 \\ \hline & 16x^2 \\ & 16x^2-64 \\ \hline & 64 \end{array}$$

From this division,

$$x^6=(x^2-4)\cdot(x^4+4x^2+16)+64$$

Divide both sides with (x^2-4)

$$\frac{x^6}{x^2-4}=\frac{\cancel{(x^2-4)}(x^4+4x^2+16)}{\cancel{(x^2-4)}}+\frac{64}{x^2-4}$$

$$\frac{x^6}{x^2-4}=x^4+4x^2+16+\frac{64}{x^2-4}$$

$$=x^4+4x^2+16+\frac{64}{(x+2)(x-2)}$$ Since $a^2-b^2=(a+b)(a-b)$

$$=x^4+4x^2+16+\frac{A}{x+2}+\frac{B}{x-2}$$, The form of the partial fraction

Decomposition

Where A,B are constants

Therefore, the partial decomposition is $\boxed{\frac{x^6}{x^2-4}=x^4+4x^2+16+\frac{A}{x+2}+\frac{B}{x-2}}$.

(b)

Consider the fraction, $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2}$

It is proper fraction because the power of x in numerator is less than the power of x in denominator.

The degree of first factor of denominator is 2 and degree of second factor of denominator is 4.

So, the degree of denominator becomes 6.

Therefore, the form of the partial fraction decomposition is

$$\boxed{\frac{x^4}{(x^2-x+1)(x^2+2)^2}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2+2}+\frac{Ex+F}{(x^2+2)^2}}.$$

Answer 6E.

(a)

Consider the expression

$$\frac{t^6+1}{t^6+t^3}$$

Decompose the given fraction into partial fractions without evaluating the constants in decomposition.

The expression is of the form $\dfrac{P(x)}{Q(x)}$

In the given expression, the degree of the numerator $P(x)$ is equal to the degree of the denominator $Q(x)$. So apply long division to decompose.

$$\begin{array}{r|l} & \quad 1 \\ t^6+t^3 & t^6+1 \\ & (-) \\ & \underline{t^6+t^3)} \\ & 1-t^3 \end{array}$$

So, the partial fraction can be written as

$$\frac{t^6+1}{t^6+t^3}=1+\frac{1-t^3}{t^6+t^3}$$

This is of the form $1+\dfrac{R(x)}{Q(x)}$.

Now, try to simplify the denominator the fraction $Q(x)$.

The factored form of the denominator t^6+t^3 is $t^3(t^3+1)$

Use synthetic division to further simplify.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

Therefore, $t^3+1=(t+1)(t^2-t+1)$

Thus, the simplified denominator is $t^3(t+1)(t^2-t+1)$.

So,

$$\frac{t^6+1}{t^6+t^3}=1+\frac{1-t^3}{t^6+t^3}$$

$$=1+\frac{1-t^3}{t^3(t+1)(t^2-t+1)}$$

In the denominator $Q(x)$, $t^3(t+1)(t^2-t+1)$, the factor t is repeated thrice and one distinct linear factor $t+1$ and one irreducible factor t^2-t+1.

For the factor t, the decomposition is

$$\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}$$

For the factor $t+1$, the decomposition is

$$\frac{D}{t+1}$$

For the factor t^2-t+1,

$$\begin{aligned} b^2-4ac &= (-1)^2-4\cdot1\cdot1 \\ &= 1-4 \\ &= -3<0 \end{aligned}$$

So, the expression can be decomposed as

$$\frac{Et+C}{t^2-t+1}$$

Finally, the partial fraction decomposition of given expression is

$$\boxed{\frac{t^6+1}{t^6+t^3}=1+\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{t+1}+\frac{Et+F}{t^2-t+1}}.$$

Where *A, B, C, D, E*, and *F* are constants.

(b)

Consider the expression

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$$

Decompose the given fraction into partial fractions without evaluating the constants in decomposition.

The expression is of the form $\frac{P(x)}{Q(x)}$

In the given expression, the degree of the numerator $P(x)$ is less than the degree of the denominator $Q(x)$. so, the rational fraction is proper.

Now, try to simplify the denominator the fraction $Q(x)$.

The factored form of the denominator terms are x^2-x as $x(x-1)$ and x^4+2x^2+1 as $(x^2+1)^2$

Thus, the simplified denominator is $x(x-1)(x^2+1)^2$.

So,

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)}=\frac{x^5+1}{x(x-1)(x^2+1)^2}$$

In the denominator $Q(x)$, $x(x-1)(x^2+1)^2$, x is one distinct factor and $x-1$ isone distinct linear factor and one irreducible factor repeated factor $(x^2+1)^2$.

For the factor x, the decomposition is

$$\frac{A}{x}$$

For the factor $x-1$, the decomposition is

$$\frac{B}{x-1}$$

For the factor $\left(x^2+1\right)^2$,

$$\begin{aligned} b^2-4ac &= (0)^2-4\cdot1\cdot1 \\ &= 0-4 \\ &= -4<0 \end{aligned}$$

So, the expression can be decomposed as

$$\frac{Cx+D}{\left(x^2+1\right)}+\frac{Ex+F}{\left(x^2+1\right)^2}$$

Finally, the partial fraction decomposition of given expression is

$$\boxed{\frac{x^5+1}{\left(x^2-x\right)\left(x^4+2x^2+1\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{\left(x^2+1\right)^2}}$$

Where *A*, *B*, *C*, *D*, *E*, and *F* are constants.

Answer 7E.

Consider the integral $\int \frac{x^4}{x-1}dx$

Evaluate the above integral.

Since the degree of the numerator is greater than the degree of the denominator, first perform the long division.

$$\begin{array}{r} x^3+x^2+x+1 \\ x-1\overline{\big)\quad x^4} \\ \underline{x^4-x^3} \\ x^3 \\ \underline{x^3-x^2} \\ x^2 \\ \underline{x^2-x} \\ x \\ \underline{x-1} \\ 1 \end{array}$$

The integral can be written as follows:

$$\begin{aligned} \int \frac{x^4}{x-1}dx &= \int\left(x^3+x^2+x+1+\frac{1}{x-1}\right) \\ &= \frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|+C \quad \left[\int x^n dx=\frac{x^{n+1}}{n+1}+C \;\&\; \int\frac{1}{x}dx=\ln|x|+C\right] \end{aligned}$$

Therefore, $\int \frac{x^4}{x-1}dx = \boxed{\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|+C}$

Answer 8E.

Consider the integral $\int \frac{3t-2}{t+1}dt$.

Simplify the term.

$$\frac{3t-2}{t+1} = \frac{3t-2+3-3}{t+1} \quad \text{Add and subtract with the number 3}$$

$$= \frac{3t+3-2-3}{t+1}$$

$$= \frac{3(t+1)-5}{t+1}$$

$$= \frac{3\cancel{(t+1)}}{\cancel{t+1}} - \frac{5}{t+1}$$

$$= 3 - \frac{5}{t+1}$$

Evaluate the integral $\int \frac{3t-2}{t+1}dt$.

$$\int \frac{3t-2}{t+1}dt = \int\left(3 - \frac{5}{t+1}\right)dt$$

$$= \int 3dt - \int \frac{5}{t+1}dt$$

$$= 3t - 5\ln|t+1| + C \qquad \text{Since } \int \frac{1}{x} = \ln|x| + C$$

Hence, $\boxed{\int \frac{3t-2}{t+1}dt = 3t - 5\ln|t+1| + C}$

Answer 9E.

Given $\int \frac{5x+1}{(2x+1)(x-1)}dx$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A(x-1)}{(2x+1)(x-1)} + \frac{B(2x+1)}{(2x+1)(x-1)}$$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A(x-1)+B(2x+1)}{(2x+1)(x-1)}$$

Now comparing both sides we get

$\Rightarrow \quad A(x-1)+B(2x+1) = 5x+1$

$\Rightarrow \quad Ax - A + 2Bx + B = 5x+1$

$\Rightarrow \quad A+2B = 5, -A+B = 1$

Now solving we get $A = 1, B = 2$

Therefore $\frac{5x+1}{(2x+1)(x-1)} = \frac{1}{2x+1} + \frac{2}{x-1}$

$$\int \frac{5x+1}{(2x+1)(x-1)}dx = \int \frac{1}{2x+1}dx + 2\int \frac{1}{x-1}dx$$

$$= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + c$$

Answer 10E.

Apply partial fraction decomposition and solve for the coefficients

$$\frac{y}{(y+4)(2y-1)}=\frac{A}{y+4}+\frac{B}{2y-1} \qquad \text{......(1)}$$

$$\frac{y}{(y+4)(2y-1)}=\frac{A(2y-1)}{(y+4)(2y-1)}+\frac{B(y+4)}{(y+4)(2y-1)}$$

$$\frac{y}{(y+4)(2y-1)}=\frac{A(2y-1)+B(y+4)}{(y+4)(2y-1)}$$

Now comparing on both sides get

$$A(2y-1)+B(y+4)=y$$
$$(2A+B)y+(-A+4B)=y$$

Compare the coefficients of like terms

$$1=2A+B \qquad \text{(coefficient of } y)$$
$$0=-A+4B \qquad \text{(constant term)}$$

Solve the above equations for A and B.

$$A=\frac{4}{9}, B=\frac{1}{9}$$

Substitute the values of A and B in (1)

$$\frac{y}{(y+4)(2y-1)}=\frac{\frac{4}{9}}{y+4}+\frac{\frac{1}{9}}{2y-1}$$
$$=\frac{4}{9}\cdot\frac{1}{y+4}+\frac{1}{9}\cdot\frac{1}{2y-1}$$

Apply the integral on both sides

$$\int\left(\frac{y}{(y+4)(2y-1)}\right)dy=\int\left(\frac{4}{9}\cdot\frac{1}{y+4}+\frac{1}{9}\cdot\frac{1}{2y-1}\right)dy$$
$$=\int\frac{4}{9}\cdot\frac{1}{y+4}dy+\int\frac{1}{9}\cdot\frac{1}{2y-1}dy$$
$$=\frac{4}{9}\cdot\int\frac{1}{y+4}dy+\frac{1}{9}\cdot\int\frac{1}{2y-1}dy \qquad \left[\int\frac{1}{x}dx=\ln|x|+C\right]$$
$$=\frac{4}{9}\ln|y+4|+\frac{1}{18}\ln|2y-1|+C$$

Therefore,

$$\int\left(\frac{y}{(y+4)(2y-1)}\right)dy=\boxed{\frac{4}{9}\ln|y+4|+\frac{1}{18}\ln|2y-1|+C}.$$

Answer 11E.

Given $\int_0^1\frac{2}{2x^2+3x+1}dx$

$$\frac{2}{2x^2+3x+1}=\frac{2}{(2x+1)(x+1)}$$
$$=\frac{A}{2x+1}+\frac{B}{x+1}$$

$$\frac{2}{2x^2+3x+1}=\frac{A(x+1)}{(x+1)(2x+1)}+\frac{B(2x+1)}{(x+1)(2x+1)}$$

$$\frac{2}{2x^2+3x+1}=\frac{A(x+1)+B(2x+1)}{(x+1)(2x+1)}$$

Now comparing on both sides we get

$$\Rightarrow \quad A(x+1)+B(2x+1)=2$$
$$\Rightarrow \quad (A+2B)x+(A+B)=2$$

Now comparing coefficients on both sides we get

$\Rightarrow \quad A+2B=0, A+B=2$

Now solving we get $A=4, B=-2$

$$\int_0^1 \frac{2}{2x^2+3x+1}dx = \int_0^1 \left[\frac{4}{2x+1}+\frac{-2}{x+1}\right]dx$$

$$= \left[\frac{4}{2}\ln|2x+1| - 2\ln|x+1|\right]_0^1$$

$$= 2\left[\ln\left|\frac{2x+1}{x+1}\right|\right]_0^1$$

$$= 2\left[\ln\frac{3}{2} - \ln 1\right]$$

$$= 2\ln\frac{3}{2}$$

Answer 12E.

Given $\int_0^1 \frac{x-4}{x^2-5x+6}dx$

$$\frac{x-4}{x^2-5x+6} = \frac{A}{x-2}+\frac{B}{x-3}$$

$$\frac{x-4}{x^2-5x+6} = \frac{A(x-3)}{(x-2)(x-3)}+\frac{B(x-2)}{(x-2)(x-3)}$$

$$\frac{x-4}{x^2-5x+6} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

Now comparing on both sides we get

$\Rightarrow \quad A(x-3)+B(x-2) = x-4$

$\Rightarrow \quad (A+B)x-(3A+2B) = x-4$

Now comparing coefficients on both sides we get

$\Rightarrow A+B=1,\ 3A+2B=4$

Now solving we get $A=2, B=-1$

Therefore $\int_0^1 \frac{x-4}{x^2-5x+6}dx = \int_0^1 \left(\frac{2}{x-2}-\frac{1}{x-3}\right)dx$

$$= \left[2\ln|x-2| - \ln|x-3|\right]_0^1$$

$$= 2\ln 1 - \ln 2 - 2\ln 2 + \ln 3$$

$$= -3\ln 2 + \ln 3$$

$$= \ln 3 - 3\ln 2$$

Answer 13E.

To evaluate the integral $\int \frac{ax}{x^2-bx}dx$

$$\int \frac{ax}{x^2-bx}dx = \int \frac{ax}{x(x-b)}dx$$

$$= \int \frac{a}{x-b}dx$$

$$= a\int \frac{1}{x-b}dx$$

$= a\ln|x-b|+C$ since $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$

Hence, $\int \frac{ax}{x^2-bx}dx = \boxed{a\ln|x-b|+C}$

Answer 14E.

We have to evaluate $\int \frac{1}{(x+a)(x+b)} dx$.

The method of partial fractions gives

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

And therefore $1 = A(x+b) + B(x+a)$

If we Put $x = -a$ we get, $1 = A(b-a) \Rightarrow A = \frac{1}{b-a}$

If we Put $x = -b$ we get, $1 = B(-b+a) \Rightarrow B = \frac{1}{a-b}$

Thus $\frac{1}{(x+a)(x+b)} = \frac{\left(\frac{1}{b-a}\right)}{(x+a)} + \frac{\left(\frac{1}{a-b}\right)}{(x+b)}$

$$= \frac{1}{b-a} \cdot \frac{1}{(x+a)} + \frac{1}{a-b} \cdot \frac{1}{(x+b)}$$

Therefore $\int \frac{1}{(x+a)(x+b)} dx = \int \left(\frac{1}{b-a} \cdot \frac{1}{x+a} + \frac{1}{a-b} \cdot \frac{1}{x+b} \right) dx$

$$= \frac{1}{b-a} \ln|x+a| + \frac{1}{a-b} \ln|x+b| + C$$

$$= \frac{1}{b-a} \ln|x+a| - \frac{1}{b-a} \ln|x+b| + C$$

$$= \frac{1}{b-a} \ln\left|\frac{x+a}{x+b}\right| + C$$

Hence $\boxed{\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln\left|\frac{x+a}{x+b}\right| + C}$.

Answer 15E.

Consider the integral $\int_3^4 \left(\frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} \right) dx$.

The objective is to find the integral.

Rewrite the integral as,

$$\int_3^4 \left(\frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} \right) dx = \int_3^4 \left(\frac{x^3 - 2x^2}{x^3 - 2x^2} - \frac{4}{x^3 - 2x^2} \right) dx$$

$$= \int_3^4 \left(1 - \frac{4}{x^3 - 2x^2} \right) dx \qquad \text{......(1)}$$

To solve the integral, first we need to write the decomposition of the fraction $\frac{4}{x^3-2x^2}$.

The partial decomposition of the fraction is,

$$\frac{4}{x^3-2x^2}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-2}$$
$$\frac{4}{x^3-2x^2}=\frac{A(x-2)+Bx(x-2)+C(x^2)}{x^2(x-2)} \quad \left(\text{Since}, x^3-2x^2=x^2(x-2)\right)$$
$$4=A(x-2)+B(x^2-2x)+C(x^2)$$
$$4=(B+C)x^2+(A-2B)x-2A$$

Compare x^2 coefficients on both sides,

$$B+C=0$$

Compare x coefficient on both sides,

$$A-2B=0$$

Compare constant coefficient on both sides,

$$-2A=4$$
$$A=\frac{4}{-2}$$
$$=-2$$

Substitute this value into the equation $A-2B=0$.

$$A-2B=0$$
$$-2-2B=0$$
$$B=\frac{-2}{2}$$
$$=-1$$

Substitute this value into the equation $B+C=0$.

$$B+C=0$$
$$-1+C=0$$
$$C=1$$

The partial fraction is,

$$\frac{4}{x^3-2x^2}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-2}$$
$$=\frac{1}{x-2}-\frac{2}{x^2}-\frac{1}{x}$$

Hence, the partial decomposition of the fraction is $\boxed{\frac{4}{x^3-2x^2}=\frac{1}{x-2}-\frac{2}{x^2}-\frac{1}{x}}$.

Rewrite the equation (1) as,

$$\int_3^4\left(\frac{x^3-2x^2-4}{x^3-2x^2}\right)dx=\int_3^4\left(\frac{x^3-2x^2}{x^3-2x^2}-\frac{4}{x^3-2x^2}\right)dx$$
$$=\int_3^4\left(1-\frac{4}{x^3-2x^2}\right)dx$$
$$=\int_3^4\left(1-\left(\frac{1}{x-2}-\frac{2}{x^2}-\frac{1}{x}\right)\right)dx$$
$$=\int_3^4\left(1-\frac{1}{x-2}+\frac{2}{x^2}+\frac{1}{x}\right)dx$$

$$=\left(x-\ln(x-2)-\frac{2}{x}+\ln(x)\right)_3^4$$
$$=\left(4-\ln(4-2)-\frac{2}{4}+\ln(4)\right)-\left(3-\ln(3-2)-\frac{2}{3}+\ln(3)\right)$$
$$=\left(-\ln(2)+\frac{7}{2}+2\ln(2)\right)-\left(-\ln(1)+\frac{7}{3}+\ln(3)\right)$$
$$=\frac{7}{6}+\ln(2)-\ln(3)$$

Hence, the answer is $\boxed{\int_3^4\left(\frac{x^3-2x^2-4}{x^3-2x^2}\right)dx=\frac{7}{6}+\ln(2)-\ln(3)}$.

Answer 16E.

To evaluate the integral $\int_0^1\frac{x^3-4x-10}{x^2-x-6}dx$, use partial fractions.

Since the degree of the numerator of the integrand is greater than the degree of denominator, first perform the long division.

This enables to write the integrand as:

$$\frac{x^3-4x-10}{x^2-x-6}=x+1+\frac{3x-4}{x^2-x-6}.$$

The second step is to factor the denominator $Q(x)=x^2-x-6$ in $\frac{3x-4}{x^2-x-6}$.

$$Q(x)=x^2-x-6$$
$$=x^2-3x+2x-6$$
$$=x(x-3)+2(x-3)$$
$$=(x-3)(x+2)$$

Thus, the factors of $Q(x)$ are $(x-3)$ and $(x+2)$.

The partial fraction decomposition for $\frac{3x-4}{x^2-x-6}$ is

$$\frac{3x-4}{(x-3)(x+2)}=\frac{A}{(x-3)}+\frac{B}{(x+2)}.$$

Multiply the least common denominator $(x-3)(x+2)$.

$$(3x-4)=A(x+2)+B(x-3).$$

Plug in $x=3$ to $(3x-4)=A(x+2)+B(x-3)$.

$$3(3)-4=A(3+2)+B(3-3)$$
$$5=5A+0$$
$$A=1$$

Plug in $x=-2$ to $(3x-4)=A(x+2)+B(x-3)$.

$$3(-2)-4=A(-2+2)+B(-2-3)$$
$$-10=0A-5B$$
$$B=2$$

Thus, $\dfrac{3x-4}{(x-3)(x+2)}=\dfrac{1}{(x-3)}+\dfrac{2}{(x+2)}$.

Plug in $\dfrac{x^3-4x-10}{x^2-x-6}=x+1+\dfrac{1}{(x-3)}+\dfrac{2}{(x+2)}$ to $\int_0^1\dfrac{x^3-4x-10}{x^2-x-6}dx$.

$$\int_0^1\frac{x^3-4x-10}{x^2-x-6}dx=\int_0^1\left[x+1+\frac{1}{(x-3)}+\frac{2}{(x+2)}\right]dx$$

Use the identities, $\int x^n dx=\dfrac{x^{n+1}}{n+1}+C, \int 1dx=x+C,$ and $\int\dfrac{1}{x}dx=\ln(x)+C$.

$$\int_0^1\frac{x^3-4x-10}{x^2-x-6}dx=\left[\frac{x^2}{2}+x+\ln|x-3|+2\ln|x+2|\right]_0^1$$
$$=\left[\left(\frac{1}{2}-0\right)+(1-0)+\left(\ln|1-3|-\ln|0-3|\right)+2\left(\ln|1+2|-\ln|0+2|\right)\right]$$
$$=\frac{1}{2}+1+\left(\ln|-2|-\ln|-3|\right)+2\left(\ln|3|-\ln|2|\right)$$

Simplify the terms by applying the properties of logarithms, $\ln(a)-\ln(b)=\ln\left(\dfrac{a}{b}\right)$.

$$\int_0^1\frac{x^3-4x-10}{x^2-x-6}dx=\frac{3}{2}+\ln\left(\frac{2}{3}\right)+2\ln\left(\frac{3}{2}\right)$$
$$=\frac{3}{2}+\ln\left(\frac{2}{3}\right)+\ln\left(\frac{3}{2}\right)^2 \qquad \text{since } \ln(x^n)=n\ln x$$
$$=\frac{3}{2}+\ln\left(\frac{2}{3}\right)+\ln\left(\frac{9}{4}\right)$$
$$=\frac{3}{2}+\ln\left(\frac{2}{3}\cdot\frac{9}{4}\right) \qquad \text{since } \ln(a)+\ln(b)=\ln(ab)$$
$$=\frac{3}{2}+\ln\left(\frac{3}{2}\right)$$

Therefore, the value of $\int_0^1\dfrac{x^3-4x-10}{x^2-x-6}dx$ is $\boxed{\dfrac{3}{2}+\ln\left(\dfrac{3}{2}\right)}$.

Answer 17E.

We have to evaluate $\int_1^2\dfrac{4y^2-7y-12}{y(y+2)(y-3)}dy$

Let $\dfrac{4y^2-7y-12}{y(y+2)(y-3)}=\dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3}$

$$\Rightarrow A(y+2)(y-3)+B(y-3)y+C(y)(y+2)=4y^2-7y-12$$

Putting $y=0$, $\quad -6A=-12\Rightarrow A=2$

Putting $y=-2$, $\quad 10B=18\Rightarrow B=\dfrac{18}{10}=\dfrac{9}{5}$

Putting $y=3$ $\quad 15C=3\Rightarrow C=\dfrac{1}{5}$

Then

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}dy = \int_1^2\left(\frac{2}{y}+\frac{9}{5(y+2)}+\frac{1}{5(y-3)}\right)dy$$

$$= \left[2\ln|y|+\frac{9}{5}\ln|y+2|+\frac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2+\frac{9}{5}\ln(4)+\frac{1}{5}\ln|-1|-2\ln 1-\frac{9}{5}\ln 3-\frac{1}{5}\ln|-2|$$

$$= 2\ln 2+\frac{18}{5}\ln 2+0-0-\frac{9}{5}\ln 3-\frac{1}{5}\ln 2$$

$$= \frac{27}{5}\ln 2-\frac{9}{5}\ln 3$$

$$= \frac{9}{5}[3\ln 2-\ln 3]$$

$$= \frac{9}{5}\left(\ln\frac{2^3}{3}\right) = \boxed{\frac{9}{5}\ln\frac{8}{3}}$$

Answer 18E.

We have to evaluate $\int \frac{x^2+2x-1}{x^3-x}dx$

Since $\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x^2-1)} = \frac{x^2+2x-1}{x(x-1)(x+1)}$

Let $\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$\Rightarrow x^2+2x-1 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

Put $x=1$, $2 = B.2 \Rightarrow B=1$

Put $x=-1$, $-2 = C(2) \Rightarrow C=-1$

Put $x=0$, $-1 = A(-1) \Rightarrow A=1$

Thus $\int \frac{x^2+2x-1}{x^3-x}dx = \int\left(\frac{1}{x}+\frac{1}{x-1}-\frac{1}{x+1}\right)dx$

$$= \ln|x|+\ln|x-1|-\ln|x+1|+C$$

$$= (\ln|x|+\ln|x-1|)-\ln|x+1|+C$$

$$= \boxed{\ln\left|\frac{x(x-1)}{(x+1)}\right|+C}$$

Answer 19E.

Consider the fraction $\int \frac{x^2+1}{(x-3)(x-2)^2}dx$

The denominator $Q(x)$ is a product of linear factors, some of which are repeated:

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ in equation (1), we would use

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

In this fraction, there are three linear factors with $(x-2)$ is the repeated factor

Then write the fraction as

$$\frac{x^2+1}{(x-3)(x-2)^2}=\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

Now, solve this fraction

$$\frac{x^2+1}{(x-3)(x-2)^2}=\frac{A(x-2)^2+B(x-3)(x-2)+C(x-3)}{(x-3)(x-2)^2}$$

$$x^2+1=A(x-2)^2+B(x-3)(x-2)+C(x-3) \qquad ......(1)$$

To find the constant values,

First put $x=2$ in equation (1)

$$\begin{aligned}(2)^2+1&=A(2-2)^2+B(2-3)(2-2)+C(2-3)\\4+1&=A(0)+B(0)+C(-1)\\C&=-5\end{aligned}$$

Now, put $x=3$ in equation (1)

$$\begin{aligned}(3)^2+1&=A(3-2)^2+B(3-3)(3-2)+C(3-3)\\9+1&=A(1)+B(0)+C(0)\\A&=10\end{aligned}$$

Take

$$\begin{aligned}x^2+1&=A(x-2)^2+B(x-3)(x-2)+C(x-3)\\x^2+1&=A(x^2-4x+4)+B(x^2-5x+6)+C(x-3)\\x^2+1&=(A+B)x^2+(-4A-5B+C)x+(4A+6B-3C)\end{aligned}$$

Equate the coefficients of x^2 on both sides

$$\begin{aligned}A+B&=1\\B&=1-A\\&=1-10\\&=-9\end{aligned}$$

Therefore

$$\frac{x^2+1}{(x-3)(x-2)^2}=\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

$$\frac{x^2+1}{(x-3)(x-2)^2}=\frac{10}{x-3}+\frac{-9}{x-2}+\frac{-5}{(x-2)^2}$$

Now

$$\begin{aligned}\int\frac{x^2+1}{(x-3)(x-2)^2}dx&=\int\frac{10}{x-3}dx+\int\frac{-9}{x-2}dx+\int\frac{-5}{(x-2)^2}dx\\&=10\int\frac{1}{x-3}dx-9\int\frac{1}{x-2}dx-5\int\frac{1}{(x-2)^2}dx\end{aligned}$$

Use integral formulae

$$\int\frac{1}{f(x)}dx=\ln|f(x)|+c$$

$$\int\frac{1}{(f(x))^2}dx=-\frac{1}{f(x)}+c$$

Therefore

$$\int\frac{x^2+1}{(x-3)(x-2)^2}dx=\boxed{10\ln|x-3|-9\ln|x-2|+\frac{5}{x-2}+c}$$

Answer 20E.

Consider the integral,

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx$$

The objective is to evaluate the integral.

Since the degree of numerator is less than the degree of the denominator, the partial fraction decomposition is

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{(2x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Multiplying by the least common denominator we get

$$x^2-5x+16 = A(x-2)^2 + B(2x+1)(x-2) + C(2x+1) \qquad \text{......(1)}$$

Now equate the co-efficient

Putting $x=2$ in equation (1)

$$2^2-5\cdot2+16 = A(2-2)^2 + B(2\cdot2+1)(2-2) + C(2\cdot2+1)$$
$$4-10+16 = 5C$$
$$5C = 10$$
$$C = 2$$

Putting $x=-\frac{1}{2}$ in equation (1)

$$\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) + 16 = A\left(-\frac{1}{2}-2\right)^2 + B\left(2\left(-\frac{1}{2}\right)+1\right)\left(-\frac{1}{2}-2\right) + C\left(2\left(-\frac{1}{2}\right)+1\right)$$
$$\frac{1}{4}+\frac{5}{2}+16 = A\left(-\frac{5}{2}\right)^2 \qquad A=3$$
$$\frac{1+10+64}{4} = \frac{25}{4}A$$
$$\frac{25}{4}A = \frac{75}{4}$$

Putting $x=0$ in equation (1)

$$16 = A(0-2)^2 + B(2\cdot0+1)(0-2) + C(2\cdot0+1)$$
$$16 = 4A - 2B + C \qquad 2B=-2$$
$$16 = 4\cdot3 - 2B + 2 \qquad [\text{ Putting } A=3, C=2 \text{ }] \qquad B=-1$$
$$2B = 12+2-16$$

Therefore, the simplified integral is,

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx = \int\left(\frac{3}{(2x+1)} + \frac{-1}{(x-2)} + \frac{2}{(x-2)^2}\right)dx$$
$$= 3\int\frac{dx}{(2x+1)} - \int\frac{dx}{(x-2)} + 2\int\frac{dx}{(x-2)^2}$$
$$= \boxed{3\ln|2x+1| - \ln|x-2| + 2\left(\frac{1}{x-2}\right) + C}$$

Answer 21E.

Consider the following integral:

$$\int \frac{x^3+4}{x^2+4}dx$$

Since the degree of the numerator is not less than that of the denominator, perform long division as shown below:

$$\begin{array}{r|l} & x \\ x^2+4 & x^3+4 \\ & x^3+4x \\ & (-) \\ \hline & 4-4x \end{array}$$

Thus, $\frac{x^3+4}{x^2+4} = x + \frac{4-4x}{x^2+4}$.

Therefore, $\int \frac{x^3+4}{x^2+4}dx$ is simplified as shown below:

$$\int \frac{x^3+4}{x^2+4}dx = \int x + \frac{4-4x}{x^2+4}dx$$

$$= \int x dx + \int \frac{4}{x^2+4}dx - \int \frac{4x}{x^2+4}dx \quad = \frac{x^2}{2} + \int \frac{4}{x^2+4}dx - \int \frac{4x}{x^2+4}dx \text{ ...... (1) Use } \int x^n dx = \frac{x^{n+1}}{n+1}$$

Evaluate second and third integral in (1) separately as follows:

$$\int \frac{4}{x^2+4}dx = 4\int \frac{4}{x^2+2^2}dx$$

$$= 4 \cdot \frac{1}{2}\tan^{-1}\frac{x}{2} \quad \text{Use } \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= 2\tan^{-1}\frac{x}{2}$$

To evaluate $\int \frac{4x}{x^2+4}dx$, take $u = x^2+4$ so that $du = 2xdx$.

$$\int \frac{4x}{x^2+4}dx = 2\int \frac{2x}{x^2+4}dx$$

$$= 2\int \frac{du}{u} \quad \text{Substitute } u = x^2+4 \text{ and } du = 2xdx$$

$$= 2\ln|u| \quad \text{Use } \int \frac{1}{x}dx = \ln|x|$$

$$= 2\ln|x^2+4| \quad \text{Back substitute } u = x^2+4$$

Substitute these values in equation (1).

$$\frac{x^2}{2} + \int \frac{4}{x^2+4}dx - \int \frac{4x}{x^2+4}dx = \frac{x^2}{2} + 2\tan^{-1}\frac{x}{2} - 2\ln|x^2+4| + C$$

Here C is an integrating constant.

Therefore, $\int \frac{x^3+4}{x^2+4}dx = \boxed{\frac{x^2}{2} + 2\tan^{-1}\frac{x}{2} - 2\ln|x^2+4| + C}$.

Answer 22E.

We have to evaluate $\int \frac{dS}{S^2(S-1)^2}$

Let $\quad \frac{1}{S^2(S-1)^2} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S-1} + \frac{D}{(S-1)^2}$

So $\quad 1 = AS(S-1)^2 + B(S-1)^2 + CS^2(S-1) + DS^2$

Put $S=0$ $1=B.1 \Rightarrow B=1$

Put $S=1$ $1=D \Rightarrow D=1$

Put $S=-1$ $1=A(-1)(-2)^2+B(-2)^2+C(-1)^2(-2)+D(-1)^2$

$$1=-4A+4B-2C+D$$
$$1=-4A+4-2C+1$$
$$-4=-4A-2C$$
$$2=2A+C \quad \text{--- (1)}$$

Put $S=2$ $1=A2+B+C.4+D.4$

$$1=2A+1+4C+4$$
$$-4=2A+4C$$
$$-2=A+2C \quad \text{--- (2)}$$

From (1) $C=2-2A$ --- (3)

From (2) $\Rightarrow -2=A+4-4A \Rightarrow -6=-3A \Rightarrow A=2$

From (3) $C=2-2(2)=-2 \Rightarrow C=-2$

Thus $\displaystyle\int \frac{dS}{S^2(S-1)^2}=\int\left(\frac{2}{S}+\frac{1}{S^2}-\frac{2}{S-1}+\frac{1}{(S-1)^2}\right)dS$

$$=2\ln|S|-\frac{1}{S}-2\ln|S-1|-\frac{1}{S-1}+c$$

$$=\boxed{2\ln\left|\frac{S}{S-1}\right|-\frac{1}{S}-\frac{1}{S-1}+c}$$

Answer 23E.

We have to evaluate $\displaystyle\int \frac{10}{(x-1)(x^2+9)}dx$

Let $\displaystyle\frac{10}{(x-1)(x^2+9)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9}$

So $10=A(x^2+9)+(Bx+C)(x-1)$

Or $10=x^2(A+B)+x(C-B)+(9A-C)$

Equating coefficients of

x^2 ; $0=A+B$ --- (1)

x ; $0=C-B$ --- (2)

Constant ; $10=9A-C$ --- (3)

On adding (1) and (2), we get $0=A+C$

$$\Rightarrow -A=C$$

Put in (3) $10=9A-(-A)$

$$10=10A \Rightarrow A=1$$
$$\Rightarrow C=-1$$

Put C = -1 in (3),

Thus $0=-1-B$

$$\Rightarrow B=-1$$

Therefore $\displaystyle\int \frac{10}{(x-1)(x^2+9)}dx=\int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx$

$$=\int \frac{1}{x-1}dx-\int \frac{x}{x^2+9}dx-\int \frac{1}{x^2+9}dx$$

If we substitute $u = x^2 + 9$ in second integral,

Then $du = 2xdx \Rightarrow xdx = \dfrac{du}{2}$

So $$\int \frac{10}{(x-1)(x^2+9)}dx = \int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{u}du - \int \frac{1}{x^2+9}dx$$

$$= \ln|x-1| - \frac{1}{2}\ln|u| - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

Then $$\int \frac{10}{(x-1)(x^2+9)}dx = \boxed{\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c}$$

Answer 24E.

We have to evaluate $\int \frac{x^2 - x + 6}{x^3 + 3x}dx$

Let $$\frac{x^2 - x + 6}{x^3 + 3x} = \frac{x^2 - x + 6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

Then $$x^2 - x + 6 = A(x^2+3) + (Bx + C)x$$

Or $$x^2 - x + 6 = x^2(A+B) + xC + 3A$$

Equating coefficients of

x^2 ; $1 = A + B$

x ; $-1 = C \Rightarrow C = -1$

Constant ; $6 = 3A \Rightarrow A = 2$

As $A + B = 1 \Rightarrow B = 1 - A = 1 - 2 \Rightarrow B = -1$

Thus $$\int \frac{x^2 - x + 6}{x^3 + 3x}dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx$$

$$= \int \frac{2}{x}dx - \int \frac{x}{x^2+3}dx - \int \frac{1}{x^2+3}dx \qquad \text{--- (1)}$$

Substitute $x^2 + 3 = u \Rightarrow 2xdx = du$ in second integral

Therefore $$\int \frac{x}{x^2+3}dx = \int \frac{1}{u}\cdot\frac{du}{2}$$

$$= \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln(x^2+3)$$

Hence equation (1) becomes

$$\int \frac{x^2 - x + 6}{x^3 + 3x}dx = \boxed{2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c}$$

Answer 25E.

Consider the following integral:

$$\int \frac{4x}{x^3 + x^2 + x + 1}dx = \int \frac{4x}{(x^2+1)(x+1)}dx$$

Let $$\frac{4x}{(x^2+1)(x+1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x+1} \quad \text{......(1)}$$

$$\Rightarrow \quad (Ax + B)(x+1) + C(x^2+1) = 4x$$

If $x = -1$ then $2C = -4 \Rightarrow C = -2$

If $x = 0$ then $B + C = 0 \Rightarrow B = -C$

$= 2$

Equate the coefficients of x^2 on both sides.

$$A+C=0$$
$$A=-C$$
$$=2$$

Now substitute the values of *A*, *B*, *C* in equation (1), obtain that

$$\frac{4x}{(x^2+1)(x+1)}=\frac{2x+2}{x^2+1}-\frac{2}{x+1}$$

Therefore, the given integral can be expressed as follows:

$$\int\frac{4x}{x^3+x^2+x+1}dx=\int\frac{2x+2}{x^2+1}dx-\int\frac{2}{x+1}dx$$
$$=\int\frac{2x}{x^2+1}dx+2\int\frac{1}{x^2+1}dx-2\int\frac{1}{x+1}dx$$
$$=\ln\left|x^2+1\right|+2\tan^{-1}x-2\ln\left|x+1\right|+C$$

Thus, $\int\frac{4x}{x^3+x^2+x+1}dx=\boxed{\ln\left|x^2+1\right|+2\tan^{-1}x-2\ln\left|x+1\right|+C}$, where *C* is the integrating constant.

Answer 26E.

Consider the following integral:

$$\int\frac{x^2+x+1}{(x^2+1)^2}dx$$

The form of the partial fraction decomposition is as shown below:

$$\frac{x^2+x+1}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2} \quad \text{......} \ (1)$$
$$=\frac{(Ax+B)(x^2+1)+(Cx+D)}{(x^2+1)^2}$$
$$x^2+x+1=Ax^3+Ax+Bx^2+B+Cx+D$$
$$=Ax^3+Bx^2+(A+C)x+(B+D)$$

Equate the coefficients and get the system as follows:

$$A=0, B=1, A+C=1, B+D=1$$
$$A+C=1$$
$$0+C=1$$
$$C=1$$

And

$$B+D=1$$
$$1+D=1$$
$$D=0$$

Substitute these values in (1).

$$\frac{x^2+x+1}{(x^2+1)^2} = \frac{0(x)+1}{x^2+1} + \frac{(1)x+0}{(x^2+1)^2}$$
$$= \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$$

Simplify as follows:

$$\int \frac{x^2+x+1}{(x^2+1)^2}dx = \int\left(\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$
$$= \int \frac{1}{x^2+1}dx + \int \frac{x}{(x^2+1)^2}dx$$
$$= \int \frac{1}{x^2+1^2}dx + \int \frac{x}{(x^2+1)^2}dx$$

$= \tan^{-1} x + \int \frac{x}{(x^2+1)^2}dx$ ...... (2) Since $\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

Evaluate the integral $\int \frac{x}{(x^2+1)^2}dx$.

Take $x^2+1=u$ then $2xdx = du \Rightarrow xdx = \frac{du}{2}$.

$\int \frac{x}{(x^2+1)^2}dx = \int \frac{1}{u^2}\frac{du}{2}$ Substitute $x^2+1=u$ and $xdx = \frac{du}{2}$

$= \frac{1}{2}\int \frac{1}{u^2}du$

$= \frac{1}{2}\int u^{-2}du$

$= \frac{1}{2}\left(-\frac{1}{u}\right)$ Use $\int x^n dx = \frac{x^{n+1}}{n+1}$

$= \frac{1}{2}\left(-\frac{1}{x^2+1}\right)$ Back substitute $x^2+1=u$

$= -\frac{1}{2x^2+2}$

Thus, equation (2) can be written as follows:

$$\tan^{-1} x + \int \frac{x}{(x^2+1)^2}dx = \tan^{-1} x - \frac{1}{2x^2+2} + C$$

Here, C is an integrating constant.

Therefore, $\int \frac{x^2+x+1}{(x^2+1)^2}dx = \boxed{\tan^{-1} x - \frac{1}{2x^2+2} + C}$.

Answer 27E.

Consider the following integral:

$$\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}dx$$

The objective is to evaluate the integral.

Apply partial fraction decomposition, and then solve for the coefficients.

$$\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \quad \text{......(1)}$$

$$x^3+x^2+2x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$
$$x^3+x^2+2x+1 = Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+D$$
$$x^3+x^2+2x+1 = (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$$

Compare the coefficients on both sides of the above equation, we have

$$1 = A + C \quad ......(1)$$
$$1 = B + D \quad ......(2)$$
$$2 = 2A + C \quad ......(3)$$
$$1 = 2B + D \quad ......(4)$$

Solve (1) and (3).

$$A + C = 1$$
$$\underline{2A + C = 2}$$
$$-A = -1$$
$$\Rightarrow A = 1$$

From (1), $C = 1 - 1 = 0$

Solve (2) and (4).

$$B + D = 1$$
$$\underline{2B + D = 1}$$
$$-B = 0$$
$$\Rightarrow B = 0$$

From (2), $D = 1 - 0 = 1$

Therefore, the values are:

$$A = 1$$
$$B = 0$$
$$C = 0$$
$$D = 1$$

Substitute above values in the equation (1) and apply integration, we get

$$\int \frac{x^3 + x^2 + 2x + 1}{(x^2+1)(x^2+2)} dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+2} \right) dx$$
$$= \frac{1}{2} \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2 + (\sqrt{2})^2} dx$$
$$= \frac{1}{2} \ln(x^2+1) + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) + c$$

Therefore, $\int \frac{x^3 + x^2 + 2x + 1}{(x^2+1)(x^2+2)} dx = \boxed{\frac{1}{2} \ln(x^2+1) + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) + c}$.

Answer 28E.

We have to evaluate $\int \frac{x^2 - 2x - 1}{(x-1)^2 (x^2+1)} dx$

Let $\frac{x^2 - 2x - 1}{(x-1)^2 (x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2+1}$

So $x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

$$= A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x^2 - 2x + 1)$$
$$= (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D)$$

Equating coefficients of

x^3 ; $0 = A + C$ --- (1)

x^2 ; $1 = -A + B - 2C + D$ --- (2)

x ; $-2 = A + C - 2D \Rightarrow -2 = 0 - 2D \Rightarrow D = 1$ --- (3)

Constant ; $-1 = -A + B + D$

$\Rightarrow -1 = -A + B + 1 \Rightarrow B - A = -2$ --- (4)

Put $\begin{bmatrix} B-A=-2 \\ D=1 \end{bmatrix}$ in (2) $\Rightarrow 1=-2-2C+1 \Rightarrow 2=-2C \Rightarrow C=-1$

From (1) $C=-A \Rightarrow A=1$

From (4) $B-A=-2 \Rightarrow B-1=-2 \Rightarrow B=-1$

Therefore

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx = \int\left(\frac{1}{x-1}+\frac{-1}{(x-1)^2}+\frac{-x+1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x-1}dx - \int (x-1)^{-2}dx - \frac{1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= \ln|x-1| - \frac{(x-1)^{-1}}{-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + c$$

$$= \boxed{\ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + c}$$

Answer 29E.

Consider the integral $\int \frac{x+4}{x^2+2x+5}dx$.

Compare x^2+2x+5 with ax^2+bx+c, we get $a=1, b=2, c=5$.

To integrating a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where $b^2-4ac<0$,

We complete the square in the denominator and then make a substitution.

Now $b^2-4ac=(2)^2-4(1)(5)$

$=4-20$

$=-16<0$

Therefore, $x^2+2x+5 = x^2+2.1.x+1+4$

$=(x+1)^2+2^2$ Since $(a+b)^2=a^2+2ab+b^2$

Thus we have completed the square in the denominator.

Now given integral becomes,

$$\int \frac{x+4}{x^2+2x+5}dx = \int \frac{x+4}{(x+1)^2+2^2}dx$$

Now we take the substitution $x+1=u$

Differentiating with respect to x we get, $1=\frac{du}{dx} \Rightarrow du=dx$

Therefore

$$\int \frac{x+4}{x^2+2x+5}dx$$

$$= \int \frac{(x+1)+3}{(x+1)^2+2^2}dx$$

$= \int \frac{u+3}{u^2+2^2}du$ Since $x+1=u$

$$= \int \frac{u}{u^2+2^2}du + \int \frac{3}{u^2+2^2}du$$

Continuation of the above:

$$=\frac{1}{2}\int\frac{2u}{u^2+2^2}du+3\int\frac{1}{u^2+2^2}du$$ Multiply and divided by 2

$$=\frac{1}{2}\ln\left(u^2+4\right)+3\cdot\frac{1}{2}Tan^{-1}\left(\frac{u}{2}\right)+C$$ Since $\int\frac{2u}{u^2+4}du=\ln\left(u^2+4\right)+c$

$$\int\frac{du}{u^2+a^2}=\frac{1}{a}Tan^{-1}\left(\frac{u}{a}\right)+C$$

$$=\frac{1}{2}\ln\left(u^2+4\right)+\frac{3}{2}Tan^{-1}\left(\frac{u}{2}\right)+C$$

$$=\frac{1}{2}\ln\left((x+1)^2+4\right)+\frac{3}{2}Tan^{-1}\left(\frac{x+1}{2}\right)+C$$ Since $u=x+1$

$$=\frac{1}{2}\ln\left(x^2+2x+5\right)+\frac{3}{2}Tan^{-1}\left(\frac{x+1}{2}\right)+C$$ Since $u^2+4=x^2+2x+5$

Hence $\boxed{\int\frac{x+4}{x^2+2x+5}dx=\frac{1}{2}\ln\left(x^2+2x+5\right)+\frac{3}{2}Tan^{-1}\left(\frac{x+1}{2}\right)+C}$

Answer 30E.

Consider the integral,

$$\int\frac{3x^2+x+4}{x^4+3x^2+2}dx$$

The objective is to evaluate the integral.

Since the degree of numerator is less than the degree of the denominator, the partial fraction decomposition is

$$\frac{3x^2+x+4}{x^4+3x^2+2}=\frac{Ax+B}{\left(x^2+1\right)}+\frac{Cx+D}{\left(x^2+2\right)}$$

Multiplying by the least common denominator we get

$$3x^2+x+4=(Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right)$$

$$3x^2+x+4=Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+\text{D} \quad \text{......} \ (1)$$

Now we equate the co-efficient

Put $x=0$ in equation (1)

$$4=2B+D$$
$$D=4-2B$$

Put $x=1$ in equation (1)

$$3\cdot1^2+1+4=A\cdot1^3+2A\cdot1+B\cdot1^2+2B+C\cdot1^3+C\cdot1+D\cdot1^2+\text{D}$$
$$3+1+4=A+2A+B+2B+C+C+D+D$$
$$3A+3B+2C+2D=8$$
$$3A+3B+2C+2(4-2B)=8$$
$$3A+3B+2C+8-4B=8$$
$$3A-B+2C=0 \quad \text{......} \ (2)$$

Put $x=-1$ in equation (1)

$$3\cdot(-1)^2-1+4=-A-2A+B+2B-C-C+D+D$$
$$3-1+4=-3A+3B-2C+2D$$
$$-3A+3B-2C+2(4-2B)=6$$
$$-3A+3B-2C+8-4B=6$$
$$-3A-B-2C=6-8$$
$$-3A-B-2C=-2$$

$3A+B+2C=2$ ...... (3)

Subtract the equation (3) from (2), we get

$$3A-B+2C-3A-B-2C=0-2$$
$$-2B=-2$$
$$B=1$$

And,

$$D=4-2\cdot 1$$
$$D=2$$

From equation (2),

$$3A-B+2C=0$$
$$3A-1+2C=0$$

$3A+2C=1$ ...... (4)

Put $x=2$ in equation (1)

$$3\cdot 2^2+2+4=A\cdot 2^3+2A\cdot 2+B\cdot 2^2+2B+C\cdot 2^3+C\cdot 2+D\cdot 2^2+\text{D}$$
$$12+2+4=8A+4A+4B+2B+8C+2C+4D+D$$
$$18=12A+6B+10C+5D$$
$$18=12A+6\cdot 1+10C+5\cdot 2$$
$$18=12A+6\cdot 1+10C+5\cdot 2$$
$$18=12A+6+10C+10$$
$$12A+10C=18-6-10$$
$$12A+10C=2$$

$3A+\frac{5}{2}C=\frac{1}{2}$ ...... (5)

Subtract the equation (5) from (4), we get

$$3A+2C-3A-\frac{5}{2}C=1-\frac{1}{2}$$
$$2C-\frac{5}{2}C=\frac{1}{2}$$
$$-\frac{1}{2}C=\frac{1}{2}$$
$$C=-1$$

From equation (4),

$$3A+2(-1)=1$$
$$3A-2=1$$
$$3A=3$$
$$A=1$$

Therefore, the simplified integral is,

$$\int \frac{3x^2+x+4}{x^4+3x^2+2}dx = \int\left(\frac{1\cdot x+1}{(x^2+1)}+\frac{(-1)x+2}{(x^2+2)}\right)dx$$

$$= \int\left(\frac{x+1}{(x^2+1)}+\frac{-x+2}{(x^2+2)}\right)dx$$

$$= \int\frac{x}{(x^2+1)}dx+\int\frac{1}{(x^2+1)}dx-\int\frac{x}{(x^2+2)}dx+\int\frac{2}{(x^2+2)}dx$$

$$= \int\frac{x}{(x^2+1)}dx+\int\frac{dx}{x^2+1^2}-\int\frac{x}{(x^2+2)}dx+2\int\frac{dx}{x^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2}\ln\left|x^2+1\right|+\tan^{-1}x-\frac{1}{2}\ln\left|x^2+2\right|+\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

$$= \boxed{\frac{1}{2}\ln\left|x^2+1\right|+\tan^{-1}x-\frac{1}{2}\ln\left|x^2+2\right|+\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C}.$$

Answer 31E.

Consider the integral $\int\frac{1}{x^3-1}dx$

Observe that, $\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}$

Use the method of partial fractions to resolve the expression.

$$\frac{1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

So $1=A(x^2+x+1)+(Bx+C)(x-1)$

Or $1=x^2(A+B)+x(A-B+C)+(A-C)$

Equate the coefficients on both sides, we obtain that

$x^2 \quad ; \quad 0=A+B$

$B=-A \quad \cdots\cdots(1)$

$x \quad ; \quad 0=A-B+C$

$B=A+C \quad \cdots\cdots(2)$

Constant ; $\quad 1=A-C$

$C=A-1 \quad \cdots\cdots(3)$

From equations (1) and (2), we obtain that

$$-A = A + C$$

$$C = -2A \quad \cdots\cdots(4)$$

From equations (3) and (4), we obtain that

$$A - 1 = -2A$$

$$3A = 1$$

$$A = \frac{1}{3}$$

From equation (4), we obtain that

$$C = -2\left(\frac{1}{3}\right)$$

$$= -\frac{2}{3}$$

From equation (1), we obtain that

$$B = -A$$

$$B = -\frac{1}{3}$$

Therefore,

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3(x-1)} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}$$

$$\frac{1}{x^3-1} = \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$$

Thus, the given integration is

$$\int \frac{1}{x^3-1}dx = \int\left(\frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}\right)dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{3}\int\frac{x+2}{x^2+x+1}dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{2}\times\frac{1}{3}\int\frac{2(x+2)}{x^2+x+1}dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+4}{x^2+x+1}dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+1+3}{x^2+x+1}dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx + \left(-\frac{3}{6}\right)\int\frac{1}{x^2+x+1}dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int\frac{1}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

Note that,

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

And put $u = x^2+x+1$ then $du = 2x+1$

Therefore,

$$\int \frac{1}{x^3-1}dx = \frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{6}\int \frac{du}{u} - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|u| - \frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C$$

$$= \boxed{\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C}$$

Answer 32E.

Consider the integral

$$\int_0^1 \frac{x}{x^2+4x+13}dx \quad \ldots\ldots (1)$$

It is needed to evaluate the integral.

To evaluate the integral, consider its integrand.

$$\frac{x}{x^2+4x+13}$$

Since the degree of the numerator is less than the degree of the denominator in the integrand, we don't need to divide. We write it as

$$\frac{x}{x^2+4x+13} = \frac{1}{2}\frac{2x}{x^2+4x+13}$$

$$= \frac{1}{2}\frac{2x+4-4}{x^2+4x+13}$$

$$= \frac{1}{2}\frac{2x+4}{x^2+4x+13} - \frac{1}{2}\frac{4}{x^2+4x+13}$$

$$= \frac{1}{2}\frac{2x+4}{x^2+4x+13} - 2\frac{1}{x^2+4x+13}$$

Thus

$$\frac{x}{x^2+4x+13} = \frac{1}{2}\frac{2x+4}{x^2+4x+13} - 2\frac{1}{x^2+4x+13} \quad \ldots\ldots (2)$$

From (1) and (2), we have

$$\int_0^1 \frac{x}{x^2+4x+13}dx = \int_0^1 \left[\frac{1}{2}\frac{2x+4}{x^2+4x+13} - 2\frac{1}{x^2+4x+13}\right]dx$$

$$= \frac{1}{2}\int_0^1 \frac{2x+4}{x^2+4x+13}dx - 2\int_0^1 \frac{1}{x^2+4x+13}dx \quad \ldots\ldots (3)$$

Consider the first integral in the left hand side of (3)

$$\int_0^1 \frac{2x+4}{x^2+4x+13}dx$$

Let $x^2+4x+13 = u \;\Rightarrow (2x+4)dx = du$

Then change the limits according to the substitution.

For $x=0$, $u = (0)^2+4(0)+13$

$=13$

For $x=1$, $u = (1)^2+4(1)+13$

$=18$

Then the first integral is

$$\int_0^1 \frac{2x+4}{x^2+4x+13}dx = \int_{13}^{18} \frac{1}{u}du$$

$$= \left[\ln|u|\right]_{13}^{18}$$

$$= \ln(18) - \ln(13)$$

$$= \ln\left(\frac{18}{13}\right)$$

Thus

$$\int_0^1 \frac{2x+4}{x^2+4x+13}dx = \ln\left(\frac{18}{13}\right) \text{ ...... (4)}$$

Consider the second integral in the left hand side of (3)

$$\int_0^1 \frac{1}{x^2+4x+13}dx$$

We have

$$x^2+4x+13 = x^2+2\cdot 2\cdot x+4+9$$

$$= x^2+2\cdot 2\cdot x+2^2+3^2$$

$$= (x+2)^2+3^2$$

Thus $x^2+4x+13 = (x+2)^2+3^2$

So,

$$\int_0^1 \frac{1}{x^2+4x+13}dx = \int_0^1 \frac{1}{(x+2)^2+3^2}dx$$

Let $x+2 = v$, then $dx = dv$

Then change the limits according to the substitution.

For $x = 0$, $v = 0+2$

$= 2$

For $x = 1$, $v = 1+2$

$= 3$

Then the second integral is

$$\int_0^1 \frac{1}{x^2+4x+13}dx = \int_2^3 \frac{1}{v^2+3^2}dx$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{v}{3}\right)\right]_2^3$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{3}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

$$= \frac{1}{3}\tan^{-1}(1) - \frac{1}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{2}{3}\right)\right)$$

Thus

$$\int_0^1 \frac{1}{x^2+4x+13}dx = \frac{1}{3}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{2}{3}\right)\right) \text{ ...... (5)}$$

From equation (3), we have that

$$\int_0^1 \frac{x}{x^2+4x+13}dx = \frac{1}{2}\int_0^1 \frac{2x+4}{x^2+4x+13}dx - 2\int_0^1 \frac{1}{x^2+4x+13}dx$$

$$= \frac{1}{2}\ln\left(\frac{18}{13}\right) - 2\left[\frac{1}{3}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{2}{3}\right)\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{18}{13}\right) - \frac{2}{3}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$= \frac{1}{2}\ln\left(\frac{18}{13}\right) - \frac{\pi}{6} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

Hence

$$\int_0^1 \frac{x}{x^2+4x+13}dx = \boxed{\frac{1}{2}\ln\left(\frac{18}{13}\right) - \frac{\pi}{6} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)}$$

Answer 33E.

Consider the following integral:

$$\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}dx$$

The objective is to evaluate the integral.

Use the following substitution:

Let $t = x^4 + 4x^2 + 3$, then

$$dt = 4x^3dx + 8xdx$$

$$dt = 4\left(x^3+2x\right)dx$$

$$\frac{dt}{4} = \left(x^3+2x\right)dx$$

Find the new Limits:

If $x = 0$, then,

$$t = x^4 + 4x^2 + 3$$

$$t = (0)^4 + 4(0)^2 + 3$$

$$t = 3$$

If $x = 1$, then,

$$t = x^4 + 4x^2 + 3$$

$$t = (1)^4 + 4(1)^2 + 3$$

$$t = 8$$

Substitute the above values in the given integral.

$$\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}dx = \int_3^8 \frac{1}{t}\frac{dt}{4}$$

$$= \frac{1}{4}\int_3^8 \frac{dt}{t}$$

$$= \frac{1}{4}\int_3^8 \frac{dt}{t}$$

$$= \frac{1}{4}(\ln t)_3^8$$

$$= \frac{1}{4}(\ln 8 - \ln 3)$$

$$= \frac{1}{4}\ln\left(\frac{8}{3}\right)$$

Therefore, the value of the definite integral $\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}dx$ is $\boxed{\frac{1}{4}\ln\left(\frac{8}{3}\right)}$.

Answer 34E.

$$\text{Given } \int \frac{x^5+x-1}{x^3+1}dx$$
$$= \int \left(x^2 - \frac{x^2-x+1}{x^3+1}\right)dx$$
$$= \int \left(x^2 - \frac{x^2-x+1}{(x^2-x+1)(x+1)}\right)dx$$
$$= \int \left(x^2 - \frac{1}{x+1}\right)dx$$
$$= \frac{x^3}{3} - \ln|x+1| + C$$

Answer 35E.

Consider the indefinite integral,

$$\int \frac{dx}{x(x^2+4)^2}.$$

The object is to evaluate the above integral.

Use partial fraction method to evaluate the above integral as follows.

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+4)} + \frac{Dx+E}{(x^2+4)^2} \quad \text{...... (A)}$$

This implies that,

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + x(Dx+E)$$
$$= A(x^4+8x^2+16) + (Bx^4+4Bx^2+Cx^3+4Cx) + Dx^2 + Ex$$
$$= (A+B)x^4 + Cx^3 + (8A+4B+D)x^2 + (4C+E)x + 16A$$

Equate the coefficients like powers of x on both sides.

Then,

$A+B=0$ ...... (1)

$C=0$ ...... (2)

$8A+4B+D=0$ ...... (3)

$4C+E=0$ ...... (4)

$16A=1$ ...... (5)

From equation (5), the value of A is

$$A=\frac{1}{16}.$$

Substitute $A=\frac{1}{16}$ in (1), then the value of B is,

$$B=\frac{-1}{16}$$

Substitute $A=\frac{1}{16}, B=\frac{-1}{16}$ in (3), then the value of D is,

$$\begin{aligned}8\left(\frac{1}{16}\right)+4\left(\frac{-1}{16}\right)+D&=0\\ \frac{1}{2}-\frac{1}{4}+D&=0\\ \frac{1}{4}+D&=0\\ D&=\frac{-1}{4}\end{aligned}$$

From (2) and (4), the value of E is,

$$\begin{aligned}4(0)+E&=0\\ E&=0\end{aligned}$$

Substitute all the values in equation (A), then the equation becomes,

$$\begin{aligned}\frac{1}{x\left(x^2+4\right)^2}&=\frac{1}{16x}+\frac{\frac{-1}{16}x+0}{\left(x^2+4\right)}+\frac{\frac{-1}{4}x+0}{\left(x^2+4\right)^2}\\ &=\frac{1}{16x}-\frac{x}{16\left(x^2+4\right)}-\frac{x}{4\left(x^2+4\right)^2}\end{aligned}$$

Use the above result, the integral is evaluated as follows:

$$\begin{aligned}\int\frac{dx}{x\left(x^2+4\right)^2}&=\int\left[\frac{1}{16x}-\frac{x}{16\left(x^2+4\right)}-\frac{x}{4\left(x^2+4\right)^2}\right]dx\\ &=\frac{1}{16}\int\frac{1}{x}dx-\frac{1}{16}\int\frac{x}{x^2+4}dx-\frac{1}{4}\int\frac{x}{\left(x^2+4\right)^2}dx\end{aligned}$$

Use substitution method: Let $x^2+4=u$.

Then,

$$\begin{aligned}2xdx&=du\\ xdx&=\frac{1}{2}du\end{aligned}$$

So the integral can be reduced as,

$$\begin{aligned}&\int\frac{dx}{x\left(x^2+4\right)^2}\\ &=\frac{1}{16}\int\frac{1}{x}dx-\frac{1}{16}\int\frac{x}{x^2+4}dx-\frac{1}{4}\int\frac{x}{\left(x^2+4\right)^2}dx\\ &=\frac{1}{16}\ln(x)-\frac{1}{16}\int\frac{1}{u}\frac{1}{2}du-\frac{1}{4}\int\frac{1}{u^2}\frac{1}{2}du\\ &=\frac{1}{16}\ln(x)-\frac{1}{32}\ln(u)-\frac{1}{8}\left(\frac{u^{-1}}{-1}\right)+C \quad \text{Use } \int\frac{1}{x}dx=\ln(x) \text{ and } \int x^n dx=\frac{x^{n+1}}{n+1}\\ &=\frac{1}{16}\ln(x)-\frac{1}{32}\ln\left(x^2+4\right)+\frac{1}{8\left(x^2+4\right)}+C\end{aligned}$$

Therefore, $\int\frac{dx}{x\left(x^2+4\right)^2}=\boxed{\frac{1}{16}\ln(x)-\frac{1}{32}\ln\left(x^2+4\right)+\frac{1}{8\left(x^2+4\right)}+C}$.

Answer 36E.

Consider the following integral:

$$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}dx$$

Evaluate the integral and split the denominator into two irreducible factors.

$$\frac{x^4+3x^2+1}{x^5+5x^3+5x}=\frac{x^4+3x^2+1}{x\left(x^4+5x^2+5\right)}$$

So, the partial fraction decomposition is as follows:

$$\frac{x^4+3x^2+1}{x^5+5x^3+5x}=\frac{A}{x}+\frac{Bx^3+Cx^2+Dx+E}{x^4+5x^2+5}$$

To determine A, B, C, D and E, the following equation needs to be solved:

$$x^4+3x^2+1=A\left(x^4+5x^2+5\right)+\left(Bx^3+Cx^2+Dx+E\right)(x)$$
$$=Ax^4+5Ax^2+5A+Bx^4+Cx^3+Dx^2+Ex$$

Therefore, $x^4+3x^2+1=(A+B)x^4+Cx^3+(5A+D)x^2+Ex+5A.$

This equation produces the following system of equations:

$$A+B=1$$
$$C=0$$
$$5A+D=3$$
$$E=0$$
$$5A=1 \Rightarrow A=\frac{1}{5}$$

Solving this system produces the following values:

$$A+B=1$$
$$\frac{1}{5}+B=1$$
$$B=1-\frac{1}{5}$$
$$=\frac{4}{5}$$

Solve for D as follows:

$$5A+D=3$$
$$5\left(\frac{1}{5}\right)+D=3$$
$$D=3-1$$
$$D=2$$

Substitute the values and solve as shown below:

$$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}dx=\int \frac{\frac{1}{5}}{x}+\frac{\frac{4}{5}x^3+2x}{x^4+5x^2+5}dx$$

$$=\frac{1}{5}\int \frac{1}{x}dx+\int \frac{\frac{4}{5}x^3+2x}{x^4+5x^2+5}dx$$

$$=\frac{1}{5}\int \frac{1}{x}dx+\frac{1}{5}\int \frac{5\left(\frac{4}{5}x^3+2x\right)}{x^4+5x^2+5}dx$$ Multiply and divide by 5

$$=\frac{1}{5}\int \frac{1}{x}dx+\frac{1}{5}\int \frac{4x^3+10x}{x^4+5x^2+5}dx \quad \text{...... (1)}$$

Evaluate the integrals in (1) separately as follows:

$$\int \frac{1}{x}dx = \ln|x|$$

The second integral in (1) can be determined using the method of substitution.

Let $u = x^4 + 5x^2 + 5$, so that $du = 4x^3 + 10xdx$.

$$\int \frac{4x^3 + 10x}{x^4 + 5x^2 + 5}dx = \int \frac{1}{u}du$$ Substitute $u = x^4 + 5x^2 + 5$ and $du = 4x^3 + 10xdx$

$$= \ln|u| + C$$ Use $\int \frac{1}{x}dx = \ln|x|$

$$= \ln(x^4 + 5x^2 + 5) + C$$ Back substitute $u = x^4 + 5x^2 + 5$

So, (1) can be written as follows:

$$\frac{1}{5}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{4x^3 + 10x}{x^4 + 5x^2 + 5}dx = \frac{1}{5}\ln|x| + \frac{1}{5}\ln|x^4 + 5x^2 + 5| + C$$

$$= \frac{1}{5}\ln\left(x\left(x^4 + 5x^2 + 5\right)\right) + C$$

Therefore, $\int \frac{x^4 + 3x^2 + 1}{x^5 + 5x^3 + 5x}dx = \boxed{\frac{1}{5}\ln\left(x\left(x^4 + 5x^2 + 5\right)\right) + C}$.

Answer 37E.

Consider the integral, $\int \frac{x^2 - 3x + 7}{\left(x^2 - 4x + 6\right)^2}dx$.

The objective is to evaluate the integral.

Use partial fractions to evaluate the integral.

Let $\frac{x^2 - 3x + 7}{\left(x^2 - 4x + 6\right)^2} = \frac{Ax + B}{\left(x^2 - 4x + 6\right)} + \frac{Cx + D}{\left(x^2 - 4x + 6\right)^2}$

$$\frac{x^2 - 3x + 7}{\left(x^2 - 4x + 6\right)^2} = \frac{(Ax + B)\left(x^2 - 4x + 6\right) + Cx + D}{\left(x^2 - 4x + 6\right)^2}$$

$$x^2 - 3x + 7 = (Ax + B)\left(x^2 - 4x + 6\right) + (Cx + D)$$

Equate the x^3 terms on both sides.

$A = 0$ ...... (1)

Equate the x^2 terms on both sides.

$1 = B - 4A$ ...... (2)

Equate the x terms on both sides.

$-3 = 6A - 4B + C$ ...... (3)

Equate the constant terms on both sides.

$7 = 6B + D$ ...... (4)

Substitute $A=0$ in (2).

$$1=B-4A$$
$$1=B-4(0) \qquad (\text{Since } A=0)$$
$$B=1$$

Substitute $B=1$ in (4)

$$7=6B+D$$
$$7=6(1)+D \qquad (\text{Since } B=1)$$
$$D=1$$

Substitute $B=1$ and $D=1$ in (3).

$$-3=6A-4B+C$$
$$-3=6(0)-4(1)+C \quad (\text{Since } A=0, B=1)$$
$$-3=-4+C$$
$$C=1$$

Rewrite the integral.

$$\int\frac{x^2-3x+7}{(x^2-4x+6)^2}=\int\frac{Ax+B}{(x^2-4x+6)}+\int\frac{Cx+D}{(x^2-4x+6)^2}$$
$$=\int\frac{1}{(x^2-4x+6)}dx+\int\frac{x+1}{(x^2-4x+6)^2}dx$$
$$=\int\frac{1}{(x-2)^2+2}dx+\int\frac{x-2+3}{(x^2-4x+6)^2}dx$$
$$=\int\frac{1}{(x-2)^2+2}dx+\int\frac{x-2}{(x^2-4x+6)^2}+\int\frac{3}{(x^2-4x+6)^2}dx$$
$$\int\frac{x^2-3x+7}{(x^2-4x+6)^2}=\int\frac{1}{(x-2)^2+2}dx+\int\frac{x-2}{(x^2-4x+6)^2}+\int\frac{3}{(x^2-4x+6)^2}dx \quad \text{.......} (5)$$

Consider the integral,

$$\int\frac{1}{(x-2)^2+2}dx$$
$$\int\frac{1}{(x-2)^2+2}dx=\int\frac{1}{(x-2)^2+\left(\sqrt{2}\right)^2}dx$$
$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{(x-2)}{\sqrt{2}}+C_1 \quad \left(\text{Since } \int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) \quad \text{......} (6)$$

Another integral is,

$$\int \frac{x-2}{\left(x^2-4x+6\right)^2}$$

Let $x^2-4x+6=t$

Differentiate both sides.

$$(2x-4)dx = dt$$
$$2(x-2)dx = dt$$
$$(x-2)dx = \frac{dt}{2}$$

$$\int \frac{x-2}{\left(x^2-4x+6\right)^2}dx = \int \frac{(dt/2)}{(t)^2}dx$$
$$= \frac{1}{2}\int \frac{dt}{t^2}$$
$$= \frac{1}{2}\left(\frac{t^{-2+1}}{-2+1}\right) + C_2$$
$$= -\frac{1}{2t} + C_2$$
$$= -\frac{1}{2\left(x^2-4x+6\right)} + C_2$$

That is,

$$\int \frac{x-2}{\left(x^2-4x+6\right)^2}dx = -\frac{1}{2\left(x^2-4x+6\right)} + C_2 \quad \text{...... (7)}$$

Consider,

$$\int \frac{3}{\left(x^2-4x+6\right)^2}dx = \int \frac{3}{\left[(x-2)^2+\left(\sqrt{2}\right)^2\right]^2}dx$$

Let $x-2=\sqrt{2}\tan\theta$

Differentiate on both sides.

$$dx = \sqrt{2}\sec^2\theta d\theta$$

$$\int \frac{3}{\left(x^2-4x+6\right)^2}dx = \int \frac{3}{\left[(x-2)^2+\left(\sqrt{2}\right)^2\right]^2}dx$$
$$= \int \frac{3}{\left[\left(\sqrt{2}\tan\theta\right)^2+2\right]^2}dx$$
$$= \int \frac{3}{\left[2(\tan\theta)^2+2\right]^2}dx$$

$$= \int \frac{3 \cdot \sqrt{2} \sec^2 \theta}{\left[2 \sec^2 \theta\right]^2} d\theta$$

$$= \frac{3\sqrt{2}}{4} \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \frac{3\sqrt{2}}{4} \int \frac{1}{\sec^2 \theta} d\theta$$

$$= \frac{3\sqrt{2}}{4} \int \cos^2 \theta d\theta$$

$$= \frac{3\sqrt{2}}{4} \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{3\sqrt{2}}{8} \left[\theta + \frac{\sin 2\theta}{2}\right] + C_3$$

$$= \frac{3\sqrt{2}}{8} \left[\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{\sin 2\left(\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)\right)}{2}\right] + C_3$$

This implies $\int \frac{3}{\left(x^2-4x+6\right)^2} dx = \frac{3\sqrt{2}}{8} \left[\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{\sin 2\left(\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)\right)}{2}\right] + C_3$

Next, simplify the term, $\sin 2\left(\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)\right)$.

Let $\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) = \alpha$

$$\tan \alpha = \left(\frac{x-2}{\sqrt{2}}\right)$$

Consider the expression,

$$\begin{aligned} \sin 2\left(\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)\right) &= \sin 2\alpha \\ &= \frac{2\tan\alpha}{1+\tan^2\alpha} \\ &= \frac{2\left(\frac{x-2}{\sqrt{2}}\right)}{1+\left(\frac{x-2}{\sqrt{2}}\right)^2} \\ &= \frac{2\sqrt{2}(x-2)}{x^2-4x+6} \end{aligned}$$

Then the integral $\int \frac{3}{\left(x^2-4x+6\right)^2} dx$ becomes,

$$\begin{aligned} \int \frac{3}{\left(x^2-4x+6\right)^2} dx &= \frac{3\sqrt{2}}{8} \left[\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{\sqrt{2}(x-2)}{x^2-4x+6}\right] + C_3 \\ &= \frac{3\sqrt{2}}{8} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{6(x-2)}{8\left(x^2-4x+6\right)} + C_3 \end{aligned}$$

Required integral is,

$$\int \frac{x^2-3x+7}{\left(x^2-4x+6\right)^2} = \int \frac{1}{(x-2)^2+2}dx + \int \frac{x-2}{\left(x^2-4x+6\right)^2} + \int \frac{3}{\left(x^2-4x+6\right)^2}dx$$

Substitute (6),(7) and (8) in (5)

$$\int \frac{x^2-3x+7}{\left(x^2-4x+6\right)^2}dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{(x-2)}{\sqrt{2}} + C_1 \ - \frac{1}{2\left(x^2-4x+6\right)} + C_2$$

$$+\frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{6(x-2)}{8\left(x^2-4x+6\right)} + C_3$$

$$= \frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{1}{8}\cdot\frac{6x-16}{x^2-4x+6} + C.$$

Where C is sum of constants $\left(C_1+C_2+C_3\right)$.

Therefore, the result is $\int \frac{x^2-3x+7}{\left(x^2-4x+6\right)^2}dx = \boxed{\frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{1}{8}\cdot\frac{6x-16}{x^2-4x+6} + C}$.

Answer 38E.

Consider the integral $\int \frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2}dx$

The form of the partial decomposition is,

$$\frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{\left(x^2+2x+2\right)^2}$$

$$x^3+2x^2+3x-2 = (Ax+B)\left(x^2+2x+2\right) + Cx + D$$

$$x^3+2x^2+3x-2 = Ax^3 + (2A+B)x^2 + (2A+2B+C)x + 2B + D$$

On comparing right and left hand side,

$$\begin{aligned} A &= 1 \\ 2A+B &= 2 \\ B &= 0 \\ 2A+2B+C &= 3 \\ C &= 1 \\ 2B+D &= -2 \\ D &= -2 \end{aligned}$$

Thus,

$$I = \int \frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2}dx$$

$$= \int \left(\frac{x}{x^2+2x+2} + \frac{x-2}{\left(x^2+2x+2\right)^2}\right)dx$$

$$= \int \frac{x+1}{x^2+2x+2}dx + \int \frac{-1}{x^2+2x+2}dx + \int \frac{x+1}{\left(x^2+2x+2\right)^2}dx + \int \frac{-3}{\left(x^2+2x+2\right)^2}dx$$

$$= I_1 + I_2 + I_3 + I_4$$

Now,

$$I_1 = \int \frac{x+1}{x^2+2x+2}dx$$

$$= \int \frac{1}{u}\left(\frac{1}{2}du\right) \qquad \begin{bmatrix} u = x^2+2x+2 \\ du = 2(x+1)dx \end{bmatrix}$$

$$= \frac{1}{2}\ln\left|x^2+2x+2\right| + C_1$$

$$I_2 = -\int \frac{1}{(x+1)^2+1}dx$$

$$= -\tan^{-1}(x+1) + C_2$$

$$I_3 = \int \frac{x+1}{\left[x^2+2x+2\right]^2}dx$$

$$= \int \frac{1}{u^2}\left(\frac{1}{2}du\right)$$

$$= -\frac{1}{2u} + C_3$$

$$= -\frac{1}{2\left(x^2+2x+2\right)} + C_3$$

$$I_4 = -3\int \frac{1}{\left[(x+1)^2+1\right]}dx$$

$$= -3\int \frac{1}{\left(\tan^2\theta+1\right)^2}\sec^2\theta d\theta \qquad \begin{bmatrix} \text{Consider } x+1 = \tan\theta \\ dx = \sec^2\theta \end{bmatrix}$$

$$= -3\int \cos^2\theta d\theta$$

$$= -\frac{3}{2}\int (1+\cos 2\theta) d\theta$$

$$= -\frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta + C_4\right)$$

$$= -\frac{3}{2}\theta - \frac{3}{2}\left(\frac{1}{2}\cdot 2\sin\theta\cos\theta\right) + C_4$$

$$= -\frac{3}{2}\tan^{-1}\left(\frac{x+1}{1}\right) - \frac{3}{2}\cdot\frac{x+1}{\sqrt{x^2+2x+2}}\cdot\frac{1}{\sqrt{x^2+2x+2}} + C_4$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2\left(x^2+2x+2\right)} + C_4$$

Therefore, the evaluated integral is,

$$I = I_1 + I_2 + I_3 + I_4 \qquad \left[C = C_1 + C_2 + C_3 + C_4\right]$$

$$= \frac{1}{2}\ln\left|x^2+2x+2\right| - \tan^{-1}(x+1) - \frac{1}{2\left(x^2+2x+2\right)}$$

$$- \frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2\left(x^2+2x+2\right)} + C$$

$$= \boxed{\frac{1}{2}\ln\left|x^2+2x+2\right| - \frac{5}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2\left(x^2+2x+2\right)} + C}.$$

Answer 39E.

Given $\int \frac{\sqrt{x+1}}{x}dx$

Put $\sqrt{x+1}=t \Rightarrow \frac{1}{2\sqrt{x+1}}dx = dt$

$\Rightarrow dx = 2t\,dt$

And $x+1=t^2 \Rightarrow x=t^2-1$

$$\int \frac{\sqrt{x+1}}{x}dx = \int \frac{t}{t^2-1}2t\,dt$$

$$= 2\int \frac{t^2}{t^2-1}dt$$

$$= 2\int \frac{t^2-1+1}{t^2-1}dt$$

$$= 2\int \left(1+\frac{1}{t^2-1}\right)dt$$

$$= 2\left[t+\int \frac{1}{t^2-1}dt\right]$$

$$\therefore \frac{1}{t^2-1} = \frac{1}{(t+1)(t-1)}$$

$$= \frac{A}{t+1}+\frac{B}{t-1}$$

$\Rightarrow \quad A(t-1)+B(t+1)=1$

If $t=1$ then $2B=1 \Rightarrow B=1/2$

If $t=-1$ then $-2A=1 \Rightarrow A=-1/2$

Therefore $\int \frac{\sqrt{x+1}}{x}dx = 2t+2\int\left(\frac{-1/2}{t+1}+\frac{\frac{1}{2}}{t-1}\right)dt$

$$= 2t-(\ln|t+1|-\ln|t-1|)+C$$

$$= 2\sqrt{x+1}-\ln\left|\sqrt{x+1}+1\right|+\ln\left|\sqrt{x+1}-1\right|+C$$

Answer 40E.

Consider the following integral:

$$\int \frac{dx}{2\sqrt{x+3}+x}$$

Make a substitution to express the integrand as a rational function and then evaluate the integral.

Let $\sqrt{x+3}=t.$ Then

$x+3=t^2$

$x=t^2-3$

So, $dx = 2tdt.$

Substitute the above values in $\int \frac{dx}{2\sqrt{x+3}+x}$.

$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2tdt}{2t+t^2-3}$$
$$= \int \frac{2tdt}{t^2+2t-3}$$
$$= \int \frac{2t}{(t+3)(t-1)}dt \qquad \text{By factoring.}$$
$$= \int \left[\frac{3}{2(t+3)} + \frac{1}{2(t-1)}\right]dt \qquad \text{Use partial fractions.}$$
$$= \frac{3}{2}\ln(t+3) + \frac{1}{2}\ln(t-1) + C$$
$$= \frac{3}{2}\ln\left(\sqrt{x+3}+3\right) + \frac{1}{2}\ln\left(\sqrt{x+3}-1\right) + C \quad \text{put } t = \sqrt{x+3}.$$

Therefore, $\int \frac{dx}{2\sqrt{x+3}+x} = \boxed{\frac{3}{2}\ln\left(\sqrt{x+3}+3\right) + \frac{1}{2}\ln\left(\sqrt{x+3}-1\right) + C}$.

Answer 41E.

Given $\int \frac{dx}{x^2 + x\sqrt{x}}$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow dx = 2t\,dt$

And $x = t^2 \Rightarrow x^2 = t^4$

$$\int \frac{dx}{x^2 + x\sqrt{x}} = \int \frac{2t\,dt}{t^4 + t^3}$$
$$= 2\int \frac{1}{t^2(t+1)}dt$$

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$\Rightarrow At(t+1) + B(t+1) + Ct^2 = 1$

If $t = 0$ then $B = 1$

If $t = -1$ then $C = 1$

Equate the coefficients of t^2 on both sides

$A + C = 0 \Rightarrow A = -C$

$= -1$

Therefore $\int \frac{dx}{x^2 + x\sqrt{x}} = 2\int \left(\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right)dt$

$$= -2\ln t - \frac{2}{t} + 2\ln|t+1| + C$$
$$= -2\ln\sqrt{x} - \frac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x}+1\right) + C$$

Answer 42E.

We have to evaluate $\int_0^1 \frac{1}{1+\sqrt[3]{x}}dx$

Substitute $\sqrt[3]{x} = t$

$\Rightarrow x^{1/3} = t$

$\Rightarrow x = t^3$

$\Rightarrow dx = 3t^2 dt$

For limits when $x = 0 \Rightarrow t = 0$

And $x = 1 \Rightarrow t = 1$

Thus $\int_0^1 \frac{1}{1+\sqrt[3]{x}}dx = \int_0^1 \frac{3t^2}{1+t}dt$

$$= \int_0^1 \left(3t - 3 + \frac{3}{1+t}\right)dt$$

$$= \left[\frac{3}{2}t^2 - 3t + 3\ln(1+t)\right]_0^1$$

$$= \left[\frac{3}{2} - 3 + 3\ln 2 - 3\ln 1\right] \qquad [\ln(1) = 0]$$

$$= \boxed{3\left(\ln 2 - \frac{1}{2}\right)}$$

Answer 43E.

We have to evaluate $\int \frac{x^3}{\sqrt[3]{x^2+1}}dx = \int \frac{x^3}{(x^2+1)^{1/3}}dx$

$$= \int \frac{x^2.x}{(x^2+1)^{1/3}}dx$$

Substitute $(x^2+1)^{1/3} = u \Rightarrow x^2+1 = u^3$

$$\Rightarrow 2xdx = 3u^2du$$

Thus $\int \frac{x^3}{\sqrt[3]{x^2+1}}dx = \int \frac{(u^3-1)}{u}.\frac{3}{2}u^2du$

$$= \frac{3}{2}\int (u^3-1)u\ du$$

$$= \frac{3}{2}\int (u^4-u)du$$

$$= \frac{3}{2}\left(\frac{u^5}{5} - \frac{u^2}{2}\right) + C$$

$$= \boxed{\frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C}$$

Answer 44E.

$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}dx = \int_{1/3}^{3} \frac{\sqrt{x}}{x(x+1)}dx$$

$$= \int_{1/3}^{3} \frac{1}{\sqrt{x}(x+1)}dx$$

Substitute $\sqrt{x} = t \Rightarrow x = t^2$

$$\Rightarrow dx = 2t\,dt$$

When $x = \frac{1}{3} \Rightarrow t = \frac{1}{\sqrt{3}}$

And when $x = 3 \Rightarrow t = \sqrt{3}$

Thus $\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}dx = \int_{1/3}^{3} \frac{1}{\sqrt{x}(x+1)}dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2t\,dt}{t(t^2+1)}$

$$= 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{t^2+1}dt$$

$$= 2\left[\tan^{-1}(t)\right]_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= 2\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= 2.\frac{\pi}{6}$$

$$= \boxed{\frac{\pi}{3}}$$

Answer 45E.

We have to evaluate $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}dx$

Substitute $\sqrt[6]{x}=t \Rightarrow x^{1/6}=t$

$\Rightarrow x=t^6$

$\Rightarrow dx=6t^5dt$

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}dx=\int \frac{1}{t^3-t^2}.6t^5dt$$

$$=6\int \frac{t^5}{t^2(t-1)}dt$$

$$=6\int \frac{t^3}{t-1}dt$$

$$=6\int \frac{t^3-1+1}{t-1}dt$$

$$=6\int \frac{t^3-1}{t-1}dt+6\int \frac{1}{t-1}dt$$

$$=6\int \frac{(t-1)(t^2+t+1)}{t-1}dt+6\int \frac{1}{t-1}dt$$

$$=6\int (t^2+t+1)dt+6\int \frac{1}{t-1}dt$$

Therefore

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}dx=6\left(\frac{t^3}{3}+\frac{t^2}{2}+t\right)+6.1\text{n}|t-1|+C$$

$$=2t^3+3t^2+6t+61\text{n}|t-1|+C$$

$$=\boxed{2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+61\text{n}\left|\sqrt[6]{x}-1\right|+C}$$

Answer 46E.

Consider the integration $\int \frac{\sqrt{1+\sqrt{x}}}{x}dx$ ...... (1)

Let $\sqrt{1+\sqrt{x}}=t$

$1+\sqrt{x}=t^2$

$\sqrt{x}=t^2-1$

$x=(t^2-1)^2$

$$\frac{1}{2\sqrt{1+\sqrt{x}}}\frac{d}{dx}(1+\sqrt{x})=dt$$

$$\frac{1}{2\sqrt{1+\sqrt{x}}}\frac{1}{2\sqrt{x}}dx=dt$$

$$\frac{1}{4t(t^2-1)}dx=dt$$

$$dx=4t(t^2-1)dt$$

Substitute the above values in (1).

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}dx=\int \frac{t}{(t^2-1)^2}\left(4t(t^2-1)dt\right)$$

$$=\int \frac{4t^2(t^2-1)dt}{(t^2-1)^2}$$

$$=\int \frac{4t^2dt}{(t^2-1)}$$

$$= 4\int \frac{t^2 - 1 + 1}{(t^2 - 1)} dt$$

$$= 4\int \frac{t^2 - 1}{(t^2 - 1)} dt + 4\int \frac{1}{(t^2 - 1)} dt$$

$$= 4\int dt + 4\int \frac{1}{(t^2 - 1)} dt$$

$$= 4t + 4\frac{1}{2(1)} \ln\left|\frac{t-1}{t+1}\right|$$

$$= 4\left(\sqrt{1+\sqrt{x}}\right) + 2\ln\left|\frac{\left(\sqrt{1+\sqrt{x}}\right)-1}{\left(\sqrt{1+\sqrt{x}}\right)+1}\right| \quad \left(\text{Since } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|\right)$$

Therefore $\int \frac{\sqrt{1+\sqrt{x}}}{x} dx = \boxed{4\left(\sqrt{1+\sqrt{x}}\right) + 2\ln\left|\frac{\left(\sqrt{1+\sqrt{x}}\right)-1}{\left(\sqrt{1+\sqrt{x}}\right)+1}\right|}$

Answer 47E.

We have $\int \frac{e^{2x}}{e^{2x} + 3e^x + 2} dx = \int \frac{e^x . e^x dx}{e^{2x} + 3e^x + 2}$

Substitute $e^x = t \Rightarrow e^x dx = dt$

Thus $\int \frac{e^{2x}}{e^{2x} + 3e^x + 2} dx = \int \frac{t.dt}{t^2 + 3t + 2}$

Now $\frac{t.}{t^2 + 3t + 2} = \frac{t}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1}$ (Let)

$\Rightarrow t = A(t+1) + B(t+2)$

Put $t = -1$ $\quad -1 = B$

Put $t = -2$ $\quad -2 = -A \Rightarrow A = 2$

Therefore

$$\int \frac{e^{2x}}{e^{2x} + 3e^x + 2} dx = \int \left(\frac{2}{t+2} + \frac{-1}{t+1}\right) dt$$

$$= 2\ln|t+2| - \ln|t+1| + C$$

$$= 2\ln\left(e^x + 2\right) - \ln\left(e^x + 1\right) + C$$

$$= \ln\left(e^x + 2\right)^2 - \ln\left(e^x + 1\right) + C$$

$$= \boxed{\ln\left[\left(e^x + 2\right)^2 / \left(e^x + 1\right)\right] + C}$$

Answer 48E.

Given $\int \frac{\sin x}{\cos^2 x - 3\cos x} dx$

Put $\cos x = t \quad \Rightarrow \quad -\sin x\, dx = dt$

$$\int \frac{\sin x}{\cos^2 x - 3\cos x} dx = \int \frac{-dt}{t^2 - 3t}$$

$$= -\int \frac{1}{t(t-3)} dt$$

$$\frac{1}{t(t-3)} = \frac{A}{t} + \frac{B}{t-3}$$

$$\Rightarrow \quad A(t-3)+Bt=1$$

$$t=0 \quad \Rightarrow \quad -3A=1 \quad \Rightarrow \quad A=-1/3$$

$$t=3 \quad \Rightarrow \quad 3B=1 \Rightarrow \quad B=1/3$$

Therefore $\int \frac{\sin x}{\cos^2 x-3\cos x}dx=-\int\left(\frac{-1/3}{t}+\frac{1/3}{t-3}\right)dt$

$$=\frac{1}{3}\left[\ln|t|-\ln|t-3|\right]+C$$

$$=\frac{1}{3}\left[\ln|\cos x|-\ln|\cos x-3|\right]+C$$

Answer 49E.

Consider the following integral:

$$\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}dt$$

Evaluate the integral.

Put $\tan t=u$ so that, $\sec^2 t\,dt=du$.

Substitute these values in above integral.

$$\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}dt=\int \frac{1}{u^2+3u+2}du$$

$$=\int \frac{1}{(u+1)(u+2)}du \quad \cdots\cdots(1)$$

Use Partial Fractions as shown below:

$$\frac{1}{(u+1)(u+2)}=\frac{A}{u+1}+\frac{B}{u+2}$$

$$1=A(u+2)+B(u+1)$$

$$1=u(A+B)+(2A+B)$$

Compare to obtain the following:

$$A+B=0, \qquad 2A+B=1$$

Solve these equations to obtain the following:

$$A=1, \quad B=-1$$

Therefore, $\frac{1}{(u+1)(u+2)}=\frac{1}{u+1}-\frac{1}{u+2}$.

Simplify equation (1) as follows:

$$\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}dt=\int\left[\frac{1}{u+1}-\frac{1}{u+2}\right]du$$

$$=\ln|u+1|-\ln|u+2|+C$$

$$=\ln\left|\frac{u+1}{u+2}\right|+C$$

$$=\boxed{\ln\left|\frac{\tan t+1}{\tan t+2}\right|+C} \qquad \text{Replace } u \text{ by } \tan t.$$

Answer 50E.

Consider the integral $\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx$

Let $e^x = t$

$e^x dx = dt$

$$\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx = \int \frac{e^x}{(e^x-2)((e^x)^2+1)}dx$$
$$= \int \frac{dt}{(t-2)(t^2+1)}$$

Consider $\frac{1}{(t-2)(t^2+1)}$

The partial fraction decomposition of the form.

$$\frac{1}{(t-2)(t^2+1)} = \frac{A}{(t-2)} + \frac{Bt+C}{(t^2+1)}$$
$$\frac{1}{(t-2)(t^2+1)} = \frac{A(t^2+1)+(Bt+C)(t-2)}{(t-2)(t^2+1)}$$
$$1 = A(t^2+1)+(Bt+C)(t-2)$$

Compare the t^2 terms on both sides.

$A+B=0$ ...... (1)

Compare the t terms on both sides.

$-2B+C=0$ ...... (2)

Compare the constant terms on both sides

$A-2C=1$ ...... (3)

Multiply the equation (2) with 2 and add to the equation (3).

$$-4B+2C=0$$
$$A-2C=1$$

$A-4B=1$ ...... (4)

Subtract (4) from (1).

$$A+B=0$$
$$A-4B=1$$

$$5B=-1$$
$$B=-\frac{1}{5}$$

From (1), $A=-B$

So $A=\frac{1}{5}$

Substitute $B=-\frac{1}{5}$ in (2).

$$-2B+C=0$$
$$C=2B$$
$$C=2\left(-\frac{1}{5}\right)$$
$$C=-\frac{2}{5}$$

Therefore $\frac{1}{(t-2)(t^2+1)}=\frac{A}{(t-2)}+\frac{Bt+C}{(t^2+1)}$

$$=\frac{1}{5(t-2)}+\frac{\left(-\frac{1}{5}\right)t-\frac{2}{5}}{(t^2+1)}$$
$$=\frac{1}{5(t-2)}-\frac{t}{5(t^2+1)}-\frac{2}{5}\frac{1}{(t^2+1)}$$

Simplify

$$\int\frac{e^x}{(e^x-2)(e^{2x}+1)}dx=\int\frac{dt}{(t-2)(t^2+1)}$$
$$=\int\left(\frac{1}{5(t-2)}-\frac{t}{5(t^2+1)}-\frac{2}{5}\frac{1}{(t^2+1)}\right)dx$$
$$=\int\frac{1}{5(t-2)}dx-\int\frac{t}{5(t^2+1)}dx-\int\frac{2}{5}\frac{1}{(t^2+1)}dx$$
$$=\frac{1}{5}\ln|(t-2)|-\frac{1}{10}\ln|t^2+1|-\frac{2}{5}\tan^{-1}\left(\frac{t}{1}\right)+C$$
$$=\frac{1}{5}\ln|(e^x-2)|-\frac{1}{10}\ln|(e^x)^2+1|-\frac{2}{5}\tan^{-1}(e^x)+C$$

Therefore $\int\frac{e^x}{(e^x-2)(e^{2x}+1)}dx=\boxed{\frac{1}{5}\ln|(e^x-2)|-\frac{1}{10}\ln|(e^x)^2+1|-\frac{2}{5}\tan^{-1}(e^x)+C}$

Answer 51E.

Given $\int\frac{dx}{1+e^x}$

Put $1+e^x=t \Rightarrow e^x dx=dt$

$\Rightarrow dx=e^{-x}dt$

$e^x=t-1 \Rightarrow e^{-x}=\frac{1}{t-1}$

Therefore $dx=\frac{1}{t-1}dt$

$$\int\frac{dx}{1+e^x}=\int\frac{1}{t}\left(\frac{1}{t-1}\right)dt$$

$$\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$$

$\Rightarrow \quad A(t-1)+B(t)=1$

$t=0 \Rightarrow A=-1$

$t=1 \Rightarrow B=1$

$$\int\frac{1}{1+e^x}dx=\int\left(-\frac{1}{t}+\frac{1}{t-1}\right)dt$$
$$=\ln|t-1|-\ln t+C$$
$$=\ln e^x-\ln|1+e^x|+C$$
$$=x-\ln|1+e^x|+C$$

Answer 52E.

Given $\int \frac{\cos ht}{\sin h^2 t + \sin h^4 t} dt$

Answer 52E.

Put $\sin ht = x \quad \Rightarrow \quad \cos ht\, dt = dx$

$$\int \frac{\cos ht}{\sin h^2 t + \sin h^4 t} dt = \int \frac{1}{x^2 + x^4} dx$$

$$= \int \frac{1}{x^2(x^2+1)} dx$$

$$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$\Rightarrow \quad Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = 1$

$x = 0 \quad \Rightarrow \quad B = 1$

Equate the coefficients of x^3 on both sides $A + C = 0$

Equate the coefficients of x^2 on both sides

$\Rightarrow \quad B + D = 0 \quad \Rightarrow \quad D = -B$

$= -1$

Equate the coefficients of x on both sides

$A = 0 \quad \Rightarrow \quad C = 0$

Therefore $\int \frac{\cos ht}{\sin h^2 t + \sin h^4 t} dt = \int \frac{1}{x^2(x^2+1)} dx$

$$= \int \left(\frac{1}{x^2} - \frac{1}{x^2+1} \right) dx$$

$$= -\frac{1}{x} - \tan^{-1} x + C$$

$$= -\frac{1}{\sin ht} - \tan^{-1}(\sin ht) + C$$

Answer 53E.

We have to evaluate $\int \ln(x^2 - x + 2) dx$

Integrating by parts

Take $u = \ln(x^2 - x + 2) \qquad , dv = dx$

$du = \frac{1}{x^2 - x + 2}(2x-1)dx \qquad , v = x$

$$\int \ln(x^2 - x + 2) dx = \ln(x^2 - x + 2).x - \int \frac{2x-1}{x^2 - x + 2}.x\, dx$$

$$= \ln(x^2 - x + 2)x - I \qquad \text{--- (1)}$$

Where $I = \int \frac{2x^2 - x}{x^2 - x + 2} dx$

On dividing we get

$$I = \int \frac{2x^2 - x}{x^2 - x + 2} dx = \int \left(2 + \frac{x-4}{x^2 - x + 2} \right) dx$$

$$= \int 2dx + \frac{1}{2}\int \frac{2(x-4)}{x^2 - x + 2} dx$$

$$= \int 2dx + \frac{1}{2}\int \frac{2x-1-7}{x^2 - x + 2} dx$$

$$= \int 2dx + \frac{1}{2}\int \frac{2x-1}{x^2 - x + 2} dx - \frac{7}{2}\int \frac{1}{x^2 - x + 2} dx$$

$$= \int 2dx + \frac{1}{2}\int \frac{2x-1}{x^2 - x + 2} dx - \frac{7}{2}\int \frac{1}{x^2 - x + \frac{1}{4} + 2 - \frac{1}{4}} dx$$

$$= 2x + \frac{1}{2}\ln|x^2 - x + 2| - \frac{7}{2}\int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}} dx$$

$$= 2x + \frac{1}{2}\ln|x^2 - x + 2| - \frac{7}{2} \times \frac{2}{\sqrt{7}} \tan^{-1} \frac{x - 1/2}{\sqrt{7}/2} + C_1$$

From Equation (1)

$$\int \ln\left(x^2-x+2\right)dx = x\ln\left(x^2-x+2\right)-2x-\frac{1}{2}\ln\left|x^2-x+2\right|+\sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right)-C_1$$

$$=\boxed{\left(x-\frac{1}{2}\right)\ln\left(x^2-x+2\right)-2x+\sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right)+C}$$

Where $C=-C_1$

Answer 54E.

Consider the following integral:

$$\int x\tan^{-1}x\,dx$$

The objective is to evaluate the integral.

The formula for integral by parts is given by

$$\int u\,dv = uv-\int v\,du \quad \ldots\ldots (1)$$

Apply u substitution to have the following.

$$u=\tan^{-1}x$$
$$du=\frac{1}{1+x^2}$$
$$dv=x$$
$$v=\frac{x^2}{2}$$

Substitute above values in the equation (1) and apply integration, we get

$$\begin{aligned}\int x\tan^{-1}x\,dx &= \frac{x^2}{2}\tan^{-1}x-\int\left(\frac{x^2}{2}\right)\left(\frac{1}{x^2+1}\right)dx\\ &=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{x^2+1}dx\\ &=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx\end{aligned}$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c$$

Therefore, $\int x\tan^{-1}x\,dx=\boxed{\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c}$.

Answer 55E.

Consider the function $f(x)=\frac{1}{x^2-2x-3}$.

Use a graph of $f(x)=\frac{1}{x^2-2x-3}$ to decide whether $\int_0^2 f(x)\,dx$ is positive or negative and find value of integral by using partial fractions.

To find the $y-$intercept, let $x=0$.

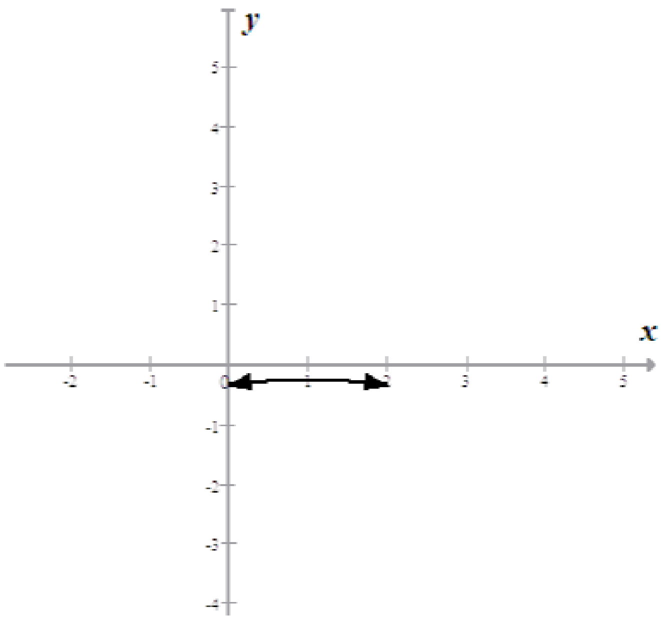
When $x=0$,

$$y=\frac{1}{0-2\cdot 0-3}$$
$$=-\frac{1}{3}, \text{ which is } y-\text{the intercept.}$$

To get different values of y, plug in different values of x within the range $0 \text{ to } 2$.

x	y
1	$-\frac{1}{4}$
$\frac{1}{2}$	$-\frac{4}{15}$
$\frac{3}{2}$	$-\frac{4}{15}$
2	$-\frac{1}{3}$

Therefore the graph is as follows



From the graph, determine that the integral $\int_0^2 f(x)dx$ is negative as the values of *y* are negative within the range $0 \text{ to } 2$ of x.

Now the value of integral is found by using partial fractions as follows

So,

$$\int_0^2 f(x)dx=\int_0^2 \frac{1}{x^2-2x-3}dx$$
$$=\int_0^2 \frac{1}{x^2-3x+x-3}dx$$
$$=\int_0^2 \frac{1}{(x-3)(x+1)}dx \qquad \text{......(1)}$$

Since the degree of numerator is less than the degree of the denominator, the partial fraction decomposition is

$$\frac{1}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}$$

Multiplying by the least common denominator we get

$$1=A(x+1)+B(x-3) \qquad \text{......}(2)$$

Now we equate the co-efficient

Putting $x=3$ in equation (2)

$$1=A(3+1)+B(3-3)$$
$$1=4A$$
$$A=\frac{1}{4}$$

Putting $x=-1$ in equation (2)

$$1=A(-1+1)+B(-1-3)$$
$$1=-4B$$
$$B=-\frac{1}{4}$$

From equation (1),

$$\int_0^2 f(x)dx=\int_0^2\frac{1}{(x-3)(x+1)}dx$$
$$=\int_0^2\left(\frac{\frac{1}{4}}{(x-3)}+\frac{-\frac{1}{4}}{(x+1)}\right)dx$$
$$=\frac{1}{4}\int_0^2\frac{1}{(x-3)}dx-\frac{1}{4}\int_0^2\frac{1}{(x+1)}dx$$
$$=\frac{1}{4}\left[\ln|x-3|\right]_0^2-\frac{1}{4}\left[\ln|x+1|\right]_0^2+C$$
$$=\frac{1}{4}\left[\ln|2-3|-\ln|0-3|\right]-\frac{1}{4}\left[\ln|2+1|-\ln|0+1|\right]+C$$
$$=\frac{1}{4}\left[\ln|-1|-\ln|-3|\right]-\frac{1}{4}\left[\ln|3|-\ln|1|\right]+C$$
$$=\frac{1}{4}\left[\ln|1|-\ln|3|\right]-\frac{1}{4}\left[\ln|3|-\ln|1|\right]+C$$
$$=\frac{1}{4}\left[0-\ln|3|\right]-\frac{1}{4}\left[\ln|3|-0\right]+C$$
$$=-\frac{1}{4}\ln|3|-\frac{1}{4}\ln|3|+C$$
$$=-\frac{2}{4}\ln|3|+C$$
$$=\boxed{-\frac{1}{2}\ln|3|+C}$$

Answer 56E.

Given $\int \frac{1}{x^2+k}dx$

Case1: $k=0$

$$\int \frac{1}{x^2+k}dx = \int \frac{1}{x^2}dx$$
$$= -\frac{1}{x}+C$$

Case2: $k>0$

$$\int \frac{1}{x^2+k}dx = \int \frac{1}{x^2+\left(\sqrt{k}\right)^2}dx$$
$$= \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{x}{\sqrt{k}}\right)+C$$

Case3: $k<0$

$$\int \frac{1}{x^2+k}dx = \int \frac{1}{x^2\left(\sqrt{k}\right)^2}dx$$
$$= \int \frac{1}{\left(x+\sqrt{k}\right)\left(x-\sqrt{k}\right)}dx$$
$$= \frac{1}{2\sqrt{k}}\left(\frac{1}{x-\sqrt{k}}-\frac{1}{x+\sqrt{k}}\right)dx$$
$$= \frac{1}{2\sqrt{k}}\left[\ln\left|x-\sqrt{k}\right|-\ln\left|x+\sqrt{k}\right|\right]+C$$

Answer 57E.

We have $\int \frac{dx}{x^2-2x} = \int \frac{dx}{x^2-2x+1-1} = \int \frac{dx}{(x-1)^2-1}$

Substitute $x-1=u \Rightarrow dx=du$

Thus $\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C$$
$$= \frac{1}{2}\ln\left|\frac{(x-1)-1}{(x-1)+1}\right|+C$$
$$= \boxed{\frac{1}{2}\ln\left|\frac{x-2}{x}\right|+C}$$

Answer 58E.

We have $\int \frac{2x+1}{4x^2+12x-7}dx = \int \frac{2x+3-2}{4x^2+12x-7}dx$

$$= \frac{1}{4}\int \frac{4(2x+3)}{4x^2+12x-7}dx - \int \frac{2}{4x^2+12x+9-16}dx$$
$$= \frac{1}{4}\int \frac{(8x+12)}{4x^2+12x-7}dx - \int \frac{2}{(2x+3)^2-16}dx$$

Let $4x^2+12x-7=t$ in first integral

and $2x+3=u$ in second integral

Then $(8x+12)\,dx=dt$ and $2dx=du$

Therefore

$$\int \frac{2x+1}{4x^2+12x-7}dx = \frac{1}{4}\int \frac{1}{t}dt - \int \frac{1}{u^2-4^2}du$$

$$= \frac{1}{4}\ln|t| - \frac{1}{2\times 4}\ln\left|\frac{u-4}{u+4}\right| + C$$

$$= \frac{1}{4}\ln\left|4x^2+12x-7\right| - \frac{1}{8}\ln\left|\frac{2x+3-4}{2x+3+4}\right| + C$$

$$= \boxed{\frac{1}{4}\ln\left|4x^2+12x-7\right| - \frac{1}{8}\ln\left|(2x-1)/(2x+7)\right| + C}$$

Answer 59E.

(A) If $t = \tan\left(\frac{x}{2}\right) \qquad -\pi < x < \pi$

Then $\tan\left(\frac{x}{2}\right) = \frac{t}{1} = \frac{\text{Altitude of right triangle}}{\text{Base of right triangle}}$

So by the Pythagorean Theorem

Hypotenuse of the right triangle $= \sqrt{(\text{base})^2 + (\text{Altitude})^2} = \sqrt{1+t^2}$

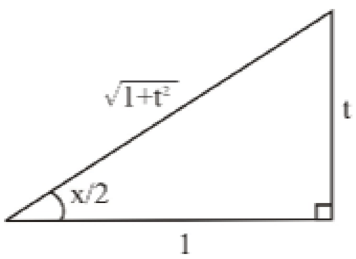


Fig. 1

Then $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}$

And $\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$

(B) We know the identity

$\cos x = 2\cos^2(x/2) - 1$ and $\sin x = 2\sin(x/2)\cos(x/2)$

Then $\cos x = 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1$ and $\sin x = 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}}$

Or $\cos x = \frac{2}{1+t^2} - 1$ and $\sin x = \frac{2t}{1+t^2}$

Or $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$

(C) As $t = \tan\left(\frac{x}{2}\right) \Rightarrow \tan^{-1} t = \frac{x}{2}$

$$\Rightarrow x = 2\tan^{-1} t$$

Differentiating both sides

$$\Rightarrow dx = \frac{2}{1+t^2}dt$$

Answer 60E.

Consider the integral

$$\int \frac{dx}{1-\cos x}$$

Take $\cos x = \dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

The integral becomes

$$\int \frac{dx}{1-\cos x} = \int \frac{dx}{1-\dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}}$$

$$= \int \frac{1+\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)-1+\tan^2\left(\frac{x}{2}\right)}dx$$

$$= \int \frac{1+\tan^2\left(\frac{x}{2}\right)}{2\tan^2\left(\frac{x}{2}\right)}dx$$

For the Trigonometric identity

$$1+\tan^2\left(\frac{x}{2}\right) = \sec^2\left(\frac{x}{2}\right)$$

$$\int \frac{dx}{1-\cos x} = \int \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)}dx$$

Let

$$\tan\left(\frac{x}{2}\right) = t$$

Then

$$\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$$

Now, the integral becomes

$$\int \frac{dx}{1-\cos x} = \int \frac{dt}{t^2}$$

$$= \int t^{-2}dt$$

$$= \frac{t^{-2+1}}{-2+1}+c$$

$$= -\frac{1}{t}+c$$

Substitute

$t=\tan\left(\frac{x}{2}\right)$

$$\int\frac{dx}{1-\cos x}=-\frac{1}{\tan\left(\frac{x}{2}\right)}+c$$

$$=-\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}+c$$

$$=-\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\frac{2\cos\left(\frac{x}{2}\right)}{2\cos\left(\frac{x}{2}\right)}+c$$

$$\left(\text{Multiply \& Divide with } 2\cos\left(\frac{x}{2}\right)\right)$$

Becomes

$$\int\frac{dx}{1-\cos x}=-\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}+c$$

$$=-\frac{(1+\cos x)}{\sin x}+c$$

$$\left(\text{Since } 2\cos^2 x=1+\cos(2x);\ 2\sin x\cos x=\sin(2x)\right)$$

$$=-\operatorname{cosec}(x)-\cot(x)+c$$

Therefore

$$\int\frac{dx}{1-\cos x}=\boxed{-\operatorname{cosec}(x)-\cot(x)+c}$$

Answer 61E.

We have to evaluate $\int\frac{1}{3\sin x-4\cos x}dx$

Let $\tan(x/2)=t$

Then $x=2\tan^{-1}t$ and $dx=\frac{2}{1+t^2}dt$

We have $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

Thus $\int\frac{1}{3\sin x-4\cos x}dx=\int\frac{1}{3.\frac{2t}{1+t^2}-4\frac{(1-t^2)}{1+t^2}}\cdot\frac{2}{1+t^2}dt$

$$=2\int\frac{1}{6t-4+4t^2}dt$$

$$=\int\frac{1}{3t-2+2t^2}dt$$

$$=\int\frac{1}{(t+2)(2t-1)}dt$$

Let $\frac{1}{(t+2)(2t-1)}=\frac{A}{(t+2)}+\frac{B}{(2t-1)}$

So $1=A(2t-1)+B(t+2)$

Put t = -2, we get A= -1/5

Put t = 1/2, we get B = 2/5

Therefore

$$\int \frac{1}{3\sin x - 4\cos x}dx = \int\left[-\frac{1}{5(t+2)}+\frac{2}{5(2t-1)}\right]dt$$

$$= \frac{1}{5}\int\left[-\frac{1}{(t+2)}+\frac{2}{(2t-1)}\right]dt$$

$$= \frac{1}{5}\left[-\ln|t+2|+\ln|2t-1|\right]+C$$

$$= \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right|+C$$

$$= \boxed{\frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right|+C}$$

Answer 62E.

We have to evaluate $\int_{\pi/3}^{\pi/2}\frac{1}{1+\sin x-\cos x}dx$

Let $\tan(x/2)=t$

Then $x=2\tan^{-1}t$ and $dx=\frac{2}{1+t^2}dt$

We have $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

For limits when $x=\frac{\pi}{3}, t=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$

And when $x=\frac{\pi}{2}, t=\tan\frac{\pi}{4}=1$

Therefore

$$\int_{\pi/3}^{\pi/2}\frac{1}{1+\sin x-\cos x}dx = \int_{1/\sqrt{3}}^{1}\frac{1}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$= \int_{1/\sqrt{3}}^{1}\frac{1+t^2}{1+t^2+2t-1+t^2}\cdot\frac{2}{1+t^2}dt$$

$$= \int_{1/\sqrt{3}}^{1}\frac{1}{t^2+t}dt$$

$$= \int_{1/\sqrt{3}}^{1}\frac{1}{t(t+1)}dt$$

Let $\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{t+1}$

So $1=A(t+1)+Bt$

Put $t=0$, we get $A=1$

Put $t=-1$, we get $B=-1$

Therefore

$$\int_{\pi/3}^{\pi/2}\frac{1}{1+\sin x-\cos x}dx = \int_{1/\sqrt{3}}^{1}\left(\frac{1}{t}-\frac{1}{t+1}\right)dt \qquad \text{[By partial fraction]}$$

$$= \left[\ln|t|-\ln|t+1|\right]_{1/\sqrt{3}}^{1}$$

$$= \left[\ln\left|\frac{t}{t+1}\right|\right]_{1/\sqrt{3}}^{1}$$

$$= \ln\left(\frac{1}{2}\right)-\ln\left(\frac{1/\sqrt{3}}{1/\sqrt{3}+1}\right)$$

$$= \ln\left(\frac{1}{2}\right)-\ln\left(\frac{1}{\sqrt{3}+1}\right)$$

$$= \boxed{\ln\left(\frac{\sqrt{3}+1}{2}\right)}$$

Answer 63E.

Consider $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}dx$ ...... (1)

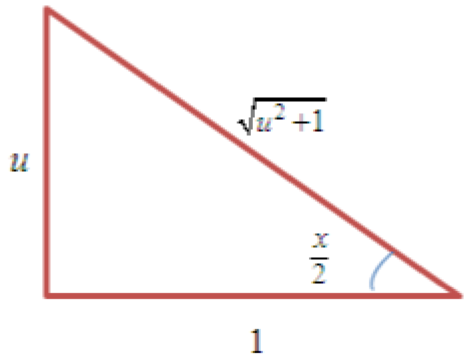
Let $u = \tan\left(\frac{x}{2}\right)$

$$\frac{x}{2} = \tan^{-1}(u)$$
$$x = 2\tan^{-1}(u)$$

$$dx = \frac{2}{1+u^2}du \text{ ...... (2)}$$

Draw the right angled triangle.

Since $u = \tan\left(\frac{x}{2}\right)$



From the above triangle

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{u^2+1}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{u}{\sqrt{u^2+1}}$$

Now find the value of $\cos x$.

$$\cos x = \cos\left(2\cdot\frac{x}{2}\right)$$
$$= 2\cos^2\left(\frac{x}{2}\right) - 1$$
$$= 2\left(\frac{1}{\sqrt{u^2+1}}\right)^2 - 1$$
$$= 2\left(\frac{1}{u^2+1}\right) - 1$$
$$= \frac{2-(u^2+1)}{u^2+1}$$
$$= \frac{1-u^2}{1+u^2}$$

Therefore $\cos x = \frac{1-u^2}{1+u^2}$ ...... (3)

Now find the value of $\sin x$.

$$\sin x = \sin\left(2 \cdot \frac{x}{2}\right)$$

$$= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$= 2\left(\frac{u}{\sqrt{u^2+1}}\right)\left(\frac{1}{\sqrt{u^2+1}}\right)$$

$$= \frac{2u}{u^2+1}$$

$$\sin x = \frac{2u}{u^2+1} \quad \text{......} \; (4)$$

Limits changes as:

If $x = 0$, then $u \to \tan\left(\frac{0}{2}\right) \to 0$.

If $x = \frac{\pi}{2}$, then $u \to \tan\left(\frac{\pi}{4}\right) \to 1$

Substitute (1), (2) and (3) in (4)

Consider $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x \cos x}{2+\cos x} dx$

$$= \int_0^1 \frac{2\left(\frac{2u}{u^2+1}\right)\left(\frac{1-u^2}{1+u^2}\right)}{2+\left(\frac{1-u^2}{1+u^2}\right)}\left(\frac{2}{1+u^2}\right) du$$

$$= \int_0^1 \frac{8\left(\frac{u\left(1-u^2\right)}{\left(u^2+1\right)^3}\right)}{\frac{2\left(1+u^2\right)+\left(1-u^2\right)}{\left(1+u^2\right)}} du$$

$$= \int_0^1 \frac{8\left(\frac{u\left(1-u^2\right)}{\left(u^2+1\right)^2}\right)}{3+u^2} du$$

$$= \int_0^1 \frac{8u\left(1-u^2\right)}{\left(3+u^2\right)\left(u^2+1\right)^2} du$$

Therefore $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x} dx = \int_0^1 \frac{8u\left(1-u^2\right)}{\left(3+u^2\right)\left(u^2+1\right)^2} du$ ...... (5)

Consider the expression for Partial fraction decomposition is

$$\frac{(1-u^2)}{(3+u^2)(u^2+1)^2}$$

Let $u^2 = t$.

$$\frac{(1-u^2)}{(3+u^2)(u^2+1)^2} = \frac{(1-t)}{(3+t)(t+1)^2}$$

$$\frac{(1-t)}{(3+t)(t+1)^2} = \frac{A}{(3+t)} + \frac{B}{(t+1)} + \frac{C}{(t+1)^2}$$

$$\frac{(1-t)}{(3+t)(t+1)^2} = \frac{A(t+1)^2 + B(3+t)(t+1)^2 + C(3+t)}{(3+t)(t+1)^2}$$

$$(1-t) = A(t+1)^2 + B(3+t)(t+1)^2 + C(3+t)$$

Let $t = -1$

$$2 = A(-1+1)^2 + B(3-1)(-1+1)^2 + C(3-1)$$
$$2 = 2C$$
$$C = 1$$

Let $t = -3$

$$(1-(-3)) = A(-3+1)^2 + B(3-3)(-3+1)^2 + C(3-3)$$
$$4 = A(-2)^2$$
$$4 = A4$$
$$A = 1$$

Let $t = 0$

$$(1-0) = A(0+1)^2 + B(3+0)(0+1)^2 + C(3+0)$$
$$1 = A + 3B + 3C$$

Substitute $A = 1$ and $C = 1$ in $1 = A + 3B + 3C$.

$$1 = A + 3B + 3C$$
$$1 = (1) + 3B + 3(1)$$
$$1 = 4 + 3B$$
$$B = -1$$

Simplify

$$\frac{(1-u^2)}{(3+u^2)(u^2+1)^2} = \frac{(1-t)}{(3+t)(t+1)^2}$$
$$= \frac{A}{(3+t)} + \frac{B}{(t+1)} + \frac{C}{(t+1)^2}$$
$$= \frac{1}{(3+t)} - \frac{1}{(t+1)} + \frac{1}{(t+1)^2}$$
$$= \frac{1}{(3+u^2)} - \frac{1}{(u^2+1)} + \frac{1}{(u^2+1)^2} \quad (\text{Since } t = u^2)$$

Form the equation (5)

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}dx = \int_0^1 \frac{8u(1-u^2)}{(3+u^2)(u^2+1)^2}du$$

$$= \int_0^1 8u\left(\frac{1}{(3+u^2)} - \frac{1}{(u^2+1)} + \frac{1}{(u^2+1)^2}\right)du$$

$$= \int_0^1 4\left(\frac{2u}{(3+u^2)}\right)du - \int_0^1 4\left(\frac{2u}{(u^2+1)}\right)du + \int_0^1 4\left(\frac{2u}{(u^2+1)^2}\right)du$$

$$= 4\left(\ln\left|(3+u^2)\right|\right)_0^1 - 4\left(\ln\left|(u^2+1)\right|\right)_0^1 - \left(\frac{4}{(u^2+1)}\right)_0^1$$

$$= 4\ln 4 - 4\ln 3 - 4\ln 2 + 4\ln 1 - 2 + 4$$
$$= 4\ln 4 - 4\ln 3 - 4\ln 2 + 2$$
$$= 4(\ln 4 - \ln 2) - 4\ln 3 + 2$$
$$= 4\ln\left(\frac{4}{2}\right) - 4\ln 3 + 2$$

$$= 4\ln(2) - 4\ln 3 + 2$$
$$= 4(\ln(2) - \ln 3) + 2$$
$$= 4\ln\left(\frac{2}{3}\right) + 2$$

Therefore $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}dx = \boxed{4\ln\left(\frac{2}{3}\right)+2}$

Answer 64E.

Consider the function $y = \frac{1}{x^3+x}$

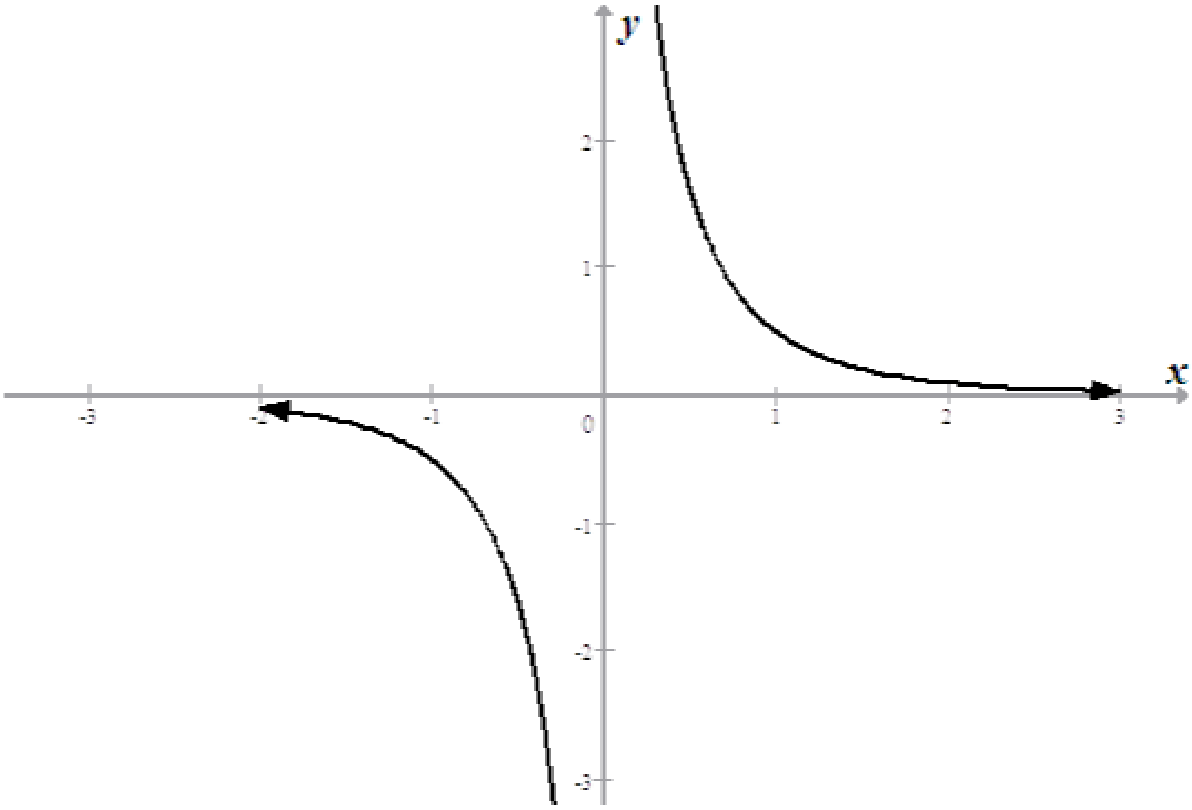
The area of region under the curve $y = \frac{1}{x^3+x}$ from $1 \text{ to } 2$ is to be found.

The curve $y = \frac{1}{x^3+x}$ is drawn as follows

By substituting, the different values of x we get the different values of y as shown in the table, which satisfies the equation.

x	y
1	$\frac{1}{2}$
0	undefined
2	$\frac{1}{10}$
-1	$-\frac{1}{2}$
-2	$-\frac{1}{10}$

Therefore the graph is as follows



Now the required area of region under the curve $y=\frac{1}{x^3+x}$ from $1 \text{ to } 2$ is $\int_1^2 y dx$

$$\int_1^2 y dx = \int_1^2 \frac{1}{x^3+x} dx$$

$$= \int_1^2 \frac{1}{x\left(x^2+1\right)} dx \qquad \text{......(1)}$$

Since the degree of numerator is less than the degree of the denominator, the partial fraction decomposition is

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiplying by the least common denominator we get

$$1 = A\left(x^2+1\right)+\left(Bx+C\right)x \qquad \text{......}(2)$$

Now we equate the co-efficient

Putting $x=0$ in equation (2)

$$1 = A(0+1)$$
$$A = 1$$

Putting $x=1$ in equation (2)

$$1 = A\left(1^2+1\right)+\left(B\cdot 1+C\right)\cdot 1$$
$$1 = 2A+B+C$$
$$1 = 2\cdot 1+B+C$$
$$B+C=-1 \qquad \text{......}(3)$$

Putting $x=-1$ in equation (2)

$$1 = A\left((-1)^2+1\right)+\left(B(-1)+C\right)(-1)$$
$$1 = 2A-(-B+C)$$
$$1 = 2\cdot 1+B-C$$
$$B-C=-1 \qquad \text{......}(4)$$

Adding equation (3) & (4) we get,

$$B+C+B-C=-1-1$$
$$2B=-2$$
$$B=-1$$

And,

$$B-C=-1$$
$$-1-C=-1$$
$$-C=0$$
$$C=0$$

From equation (1) we get,

$$\int_1^2 y dx = \int_1^2 \left(\frac{1}{x} + \frac{-1 \cdot x + 0}{(x^2+1)} \right) dx$$

Let,

$$= \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{(x^2+1)} dx \qquad \text{......}(6)$$

$$x^2 + 1 = t$$

$$2x dx = dt \qquad [\text{ Taking derivative w.r.t } dt \text{ }]$$

$$x dx = \frac{1}{2} dt$$

Now,

$$\int_1^2 \frac{x}{(x^2+1)} dx = \int_1^2 \frac{\frac{1}{2}}{t} dt$$

$$= \frac{1}{2} \int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2} \left[\ln|t| \right]_1^2 + C$$

$$= \frac{1}{2} \left[\ln|x^2+1| \right]_1^2 + C$$

$$= \frac{1}{2} \left[\ln|2^2+1| - \ln|1^2+1| \right] + C$$

$$= \frac{1}{2} \left[\ln|5| - \ln|2| \right] + C$$

From equation (6) we get,

$$\int_1^2 y dx = \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{(x^2+1)} dx$$

$$= \left[\ln|x| \right]_1^2 - \frac{1}{2} \left[\ln|5| - \ln|2| \right] + C$$

$$= \left[\ln|2| - \ln|1| \right] - \frac{1}{2} \left[\ln|5| - \ln|2| \right] + C$$

$$= \ln|2| - \ln|1| - \frac{1}{2} \ln|5| + \frac{1}{2} \ln|2| + C$$

$$= \ln|2| - \ln|1| - \frac{1}{2} \ln|5| + \frac{1}{2} \ln|2| + C$$

$$= \boxed{\frac{3}{2} \ln|2| - \frac{1}{2} \ln|5| \text{ sq.unit}}$$

Therefore $\int_1^2 \frac{1}{x^3+x} dx = \boxed{\frac{3}{2} \ln|2| - \frac{1}{2} \ln|5| \text{ sq.unit}}$

Answer 65E.

Consider the function $y = \frac{x^2+1}{3x-x^2}$ ...... (1)

Find the integration for equation (1), from the limits1 to 2.

$$\int_0^1 \left(\frac{x^2+1}{3x-x^2}\right)dx = \int_0^1 \left(\frac{x^2+1}{x(3-x)}\right)dx$$

By using long division, write the above equation as:

$$\frac{x^2+1}{x(3-x)} = -1 + \frac{3x+1}{3x-x^2} \quad \text{...... (2)}$$

Find partial fraction decomposition of the expression $\frac{3x+1}{3x-x^2}$:

$$\frac{3x+1}{3x-x^2} = \frac{3x+1}{x(3-x)}$$

$$\frac{3x+1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x}$$

$$\frac{3x+1}{x(3-x)} = \frac{A(3-x)+Bx}{x(3-x)}$$

$$3x+1 = A(3-x)+Bx$$

Let $x = 3$

$$3(3)+1 = A(3-3)+B(3)$$

$$10 = B(3)$$

$$B = \frac{10}{3}$$

Let $x = 0$

$$3(0)+1 = A(3-0)+B(0)$$

$$1 = A(3)$$

$$A = \frac{1}{3}$$

$$\frac{3x+1}{x(3-x)} = \frac{1}{3x} + \frac{10}{3(3-x)} \quad \text{...... (3)}$$

Substitute (3) in (2).

$$\frac{x^2+1}{x(3-x)} = -1 + \left(\frac{1}{3x} + \frac{10}{3(3-x)}\right)$$

Apply integration on both sides.

$$\int_1^2 \frac{x^2+1}{x(3-x)}dx = \int_1^2 (-1)dx + \int_1^2 \left(\frac{1}{3x}\right)dx + \int_1^2 \left(\frac{10}{3(3-x)}\right)dx$$

$$= (-x)_1^2 + \left(\frac{1}{3}\ln|x|\right)_1^2 - \left(\frac{10}{3}\ln|3-x|\right)_1^2$$

$$= (-2+1) + \frac{1}{3}\ln|2| - \frac{1}{3}\ln|1| - \frac{10}{3}\ln|1| + \frac{10}{3}\ln|2|$$

$$= -1 + \frac{1}{3}\ln|2| + \frac{10}{3}\ln|2|$$

$$= \ln|2|\left(\frac{1}{3} + \frac{10}{3}\right) - 1$$

$$= \ln|2|\left(\frac{11}{3}\right) - 1$$

$$= \frac{11}{3}\ln|2| - 1$$

Therefore $\int_0^1 \left(\frac{x^2+1}{3x-x^2}\right)dx = \frac{11}{3}\ln|2| - 1$

The area of the region under the curve $y = \frac{x^2+1}{3x-x^2}$ is $\boxed{\frac{11}{3}\ln|2| - 1}$

Answer 66E.

(a)

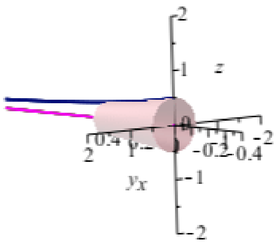
Consider the curve,

$$y = \frac{1}{(x^2+3x+2)}$$

Find the volume of the resulting solid if the region under the curve

$y = \frac{1}{(x^2+3x+2)}$ from $x = 0$ to $x = 1$ is rotated about $x-$axis.

The solid is rotated about $x-$axis is shown below:



Volume of solid can be calculated as,

$$V = \pi\int_a^b [f(x)]^2 dx$$

Use $a = 0, b = 1, y = f(x) = \frac{1}{x^2+3x+2}$

Use washer method the volume of the curve can be expressed as,

$$V = \pi\int_0^1 \left[\frac{1}{x^2+3x+2}\right]^2 dx$$

$$= \pi\int_0^1 \frac{1}{(x^2+3x+2)^2}dx$$

$$= \pi \int_0^1 \frac{1}{(x+1)(x+2)} dx$$

$$= \pi \int_0^1 \left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2} \right] dx$$

$$= \pi \left[\frac{1}{x+1} - 2\ln(x+1) + 2\ln(x+2) - \frac{1}{(x+2)} \right]_0^1$$

$$= \pi \left[\left(\frac{1}{2} - 2\ln 2 + 2\ln 3 - \frac{1}{3} \right) - \left(1 - 2\ln 1 + 2\ln 2 - \frac{1}{2} \right) \right]$$

$$= \pi \left(\frac{2}{3} - 4\ln 2 + 2\ln 3 \right)$$

Therefore, the volume of the resulting solid about $x-$axis is $\boxed{\left(\frac{2\pi}{3} - 4\pi \ln 2 + 2\pi \ln 3 \right)}$.

(b)

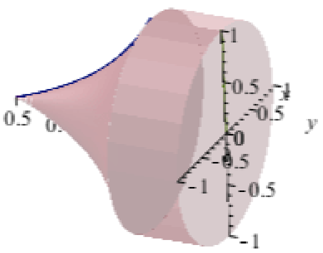
Consider the curve,

$$y = \frac{1}{(x^2 + 3x + 2)}$$

To the volume of the resulting solid if the region under the curve

$y = \frac{1}{(x^2 + 3x + 2)}$ from $x = 0$ to $x = 1$ is rotated about $y-$axis.

The solid is rotated about $y-$axis is shown below:



The volume can be obtained by using the washer method as

$$V = \int_0^1 2\pi x \left(\frac{1}{(x^2 + 3x + 2)} \right) dx$$

$$= 2\pi \int_0^1 \frac{x dx}{(x+1)(x+2)} dx$$

$$= 2\pi \int_0^1 \left(\frac{2}{x+2} - \frac{1}{x+1} \right) dx$$

$$= 2\pi \left[\int_0^1 \frac{2}{x+2} dx - \int_0^1 \frac{1}{x+1} dx \right]$$

$$= 2\pi \left[2\ln(x+2) - \ln(x+1) \right]_0^1$$

Continuation to the above step, to get

$$V = 2\pi \left[2\ln|x+2| - \ln|x+1| \right]_0^1$$

$$= 2\pi \left[(2\ln 3 - \ln 2) - (2\ln 2 - \ln 1) \right]$$

$$= 2\pi (2\ln 3 - 3\ln 2)$$

$$= 4\pi \ln 3 - 6\pi \ln 2$$

Therefore, the volume of the resulting solid about $y-$axis is $\boxed{4\pi \ln 3 - 6\pi \ln 2}$.

Answer 67E.

If P represents the number of female insects in a population, S the number of sterile males introduced each generation and r the population's natural growth rate then the female population is related to time t by $t=\int \frac{P+S}{P[(r-1)P-S]}dP$.

Consider an insect population with 10,000 females grows at a rate of $r=0.10$ and $S=900\,\text{sterile males}$.

Evaluate the integral to give an equation relating the female population to time.

To evaluate the integral, write the fraction $\frac{P+S}{P[(r-1)P-S]}$ into partial fractions.

The partial fraction of the fraction $\frac{P+S}{P[(r-1)P-S]}$ is,

$$\begin{aligned}\frac{P+S}{P[(r-1)P-S]} &= \frac{A}{P}+\frac{B}{(r-1)P-S}\\ P+S &= A[(r-1)P-S]+BP\\ &= ArP-AP-AS+BP\\ &= ((r-1)A+B)P-AS\end{aligned}$$

Compare the coefficient of S on both sides,

$$\begin{aligned}-A&=1\\ A&=-1\end{aligned}$$

Compare the coefficient of P on both sides,

$$\begin{aligned}(r-1)A+B&=1\\ -(r-1)+B&=1 \qquad (\text{Since } A=-1)\\ B&=1+r-1\\ B&=r\end{aligned}$$

The new partial fraction is $\frac{P+S}{P[(r-1)P-S]}=\frac{-1}{P}+\frac{r}{(r-1)P-S}$.

The integral becomes,

$$\begin{aligned}t&=\int \frac{P+S}{P[(r-1)P-S]}dP\\ &=\int\left(\frac{-1}{P}+\frac{r}{(r-1)P-S}\right)dP\\ &=-\int\frac{1}{P}dP+\int\frac{r}{(r-1)P-S}dP\\ &=-\int\frac{1}{P}dP+\frac{r}{r-1}\int\frac{r-1}{(r-1)P-S}dP\\ &=-\ln P+\frac{r}{r-1}\ln\left|(r-1)P-S\right|+C\end{aligned}$$

Substitute $r=0.10$ and $S=900\,\text{sterile males}$ in above equations,

$$\begin{aligned}t&=-\ln P+\frac{r}{r-1}\ln\left|(r-1)P-S\right|+C\\ &=-\ln P+\frac{0.10}{0.10-1}\ln\left|(0.10-1)P-900\right|+C\\ &=-\ln P+\frac{-1}{9}\ln\left|-0.9P-900\right|+C\\ &=-\ln P-\frac{1}{9}\ln\left|0.9P+900\right|+C \qquad (\text{Since} \left|-1\right|=1)\end{aligned}$$

Hence, the female population is related to time t is $\boxed{t=-\ln P-\frac{1}{9}\ln\left|0.9P+900\right|+C}$.

An insect population with 10,000 females grows at a rate of $r=0.10$ and 900 sterile males i.e. At $t=0$, an insect population is 10,000 females.

Substitute $P=10000$ and $t=0$ in the equation $t=-\ln P-\frac{1}{9}\ln|0.9P+900|+C$.

The equation becomes,

$$t=-\ln P-\frac{1}{9}\ln|0.9P+900|+C$$
$$=-\ln(10000)-\frac{1}{9}\ln|0.9(10000)+900|+C$$
$$=-\ln(10000)-\frac{1}{9}\ln(9900)+C$$

$$C=\ln(10000)+\frac{1}{9}\ln(9900)$$
$$=9.21+\frac{1}{9}\cdot 9.2 \qquad (\text{Since} \ln(10000)\approx 9.21 \text{ and } \ln(9900)\approx 9.2)$$
$$=9.21+1.02$$
$$=10.23$$

Hence, the female population is related to time t is $\boxed{t=-\ln P-\frac{1}{9}\ln|0.9P+900|+C}$ where $C=10.23$.

Answer 68E.

Consider the integral $\int \frac{1}{x^4+1}dx$.

To evaluate the integral, write the denominator as a difference of squares.

Rewrite the denominator as,

$$x^4+1=x^4+1+2x^2-2x^2 \qquad (\text{Add subtract with } 2x^2)$$
$$=(x^4+1+2x^2)-2x^2$$
$$=((x^2)^2+2x^2+1)-2x^2$$
$$=(x^2+1)^2-2x^2 \qquad (\text{since} (a+b)^2=a^2+2ab+b^2)$$
$$=(x^2+1)^2-(\sqrt{2}x)^2 \qquad (\text{since} (a+b)(a+b)=a^2-b^2)$$
$$=(x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)$$

Hence, the integral is $\int \frac{1}{x^4+1}dx=\int \frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}dx$.

The form of the partial fraction decomposition is,

$$\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\frac{Ax+B}{(x^2+\sqrt{2}x+1)}+\frac{Cx+D}{(x^2-\sqrt{2}x+1)}$$

Cancelling the common denominators on both sides,

$$1=(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1)$$
$$=x(x^2-\sqrt{2}x+1)A+(x^2-\sqrt{2}x+1)B+x(x^2+\sqrt{2}x+1)C+(x^2+\sqrt{2}x+1)D$$
$$=(A+C)x^3+(-\sqrt{2}A+B+\sqrt{2}C+D)x^2+(A-\sqrt{2}B+C+\sqrt{2}D)x+B+D$$

Compare x^3 coefficients on both sides,

$$A+C=0 \qquad (\text{Simplify})$$
$$A=-C$$

Compare constant terms on both sides,

$$B+D=1$$
$$D=1-B$$

Compare x coefficients on both sides,

$$A-\sqrt{2}B+C+\sqrt{2}D=0$$

$$(A+C)-\sqrt{2}(B-D)=0 \quad (\text{Since } A=-C \text{ and } D=1-B)$$

$$-\sqrt{2}(B-1+B)=0 \quad \left(\text{Divide by} \sqrt{2} \text{ on both sides}\right)$$

$$(2B-1)=0 \quad (\text{Simplify})$$

$$B=\frac{1}{2}$$

Compare x^2 coefficients on both sides,

$$-\sqrt{2}A+B+\sqrt{2}C+D=0$$

$$\sqrt{2}(C-A)+(B+D)=0 \quad \left(\text{Since } A=-C \text{ and } B=D=\frac{1}{2}\right)$$

$$\sqrt{2}(C+C)+\left(\frac{1}{2}+\frac{1}{2}\right)=0$$

$$2\sqrt{2}C=-1$$

$$C=\frac{-1}{2\sqrt{2}}$$

$$=\frac{-1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{-\sqrt{2}}{4}$$

The values are $A=\frac{\sqrt{2}}{4}, B=\frac{1}{2}, C=\frac{-\sqrt{2}}{4} \text{ and } D=\frac{1}{2}$.

The partial fraction of the fraction is,

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}=\frac{Ax+B}{\left(x^2+\sqrt{2}x+1\right)}+\frac{Cx+D}{\left(x^2-\sqrt{2}x+1\right)}$$

$$=\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{\left(x^2+\sqrt{2}x+1\right)}+\frac{\frac{-\sqrt{2}}{4}x+\frac{1}{2}}{\left(x^2-\sqrt{2}x+1\right)}$$

$$=\frac{\frac{\sqrt{2}x+2}{4}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{\frac{\sqrt{2}x-2}{4}}{\left(x^2-\sqrt{2}x+1\right)}$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{2x+2\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-2\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$

$$=\frac{\sqrt{2}}{8}\left(\frac{2x+2\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-2\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right) \quad ......(1)$$

Write the numerators $2x+2\sqrt{2}$ as $2x+\sqrt{2}+\sqrt{2}$ and denominator $2x-2\sqrt{2}$ as $2x-\sqrt{2}-\sqrt{2}$.

The equation – (1) becomes,

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}=\frac{\sqrt{2}}{8}\left(\frac{2x+2\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-2\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$

$$=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$

$$=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}+\frac{\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}+\frac{\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$

$$=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)+\frac{\sqrt{2}}{8}\left(\frac{\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}+\frac{\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$

$$=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)+\frac{1}{4}\left(\frac{1}{\left(x^2+\sqrt{2}x+1\right)}+\frac{1}{\left(x^2-\sqrt{2}x+1\right)}\right) \quad \text{......(2)}$$

Rewrite the denominator $x^2+\sqrt{2}x+1$ as,

$$x^2+\sqrt{2}x+1=(x)^2+2\cdot x\cdot\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2}$$

$$=(x)^2+2\cdot x\cdot\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

$$=\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

Rewrite the denominator $x^2-\sqrt{2}x+1$ as,

$$x^2-\sqrt{2}x+1=(x)^2-2\cdot x\cdot\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2}$$

$$=(x)^2-2\cdot x\cdot\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

$$=\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

Substitute these two denominators in equation – (2),

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$
$$+\frac{1}{4}\left(\frac{1}{\left(x^2+\sqrt{2}x+1\right)}+\frac{1}{\left(x^2-\sqrt{2}x+1\right)}\right)$$
$$=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)$$
$$+\frac{1}{4}\left(\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right)$$

The value of integral $\int\frac{1}{x^4+1}dx$ is,

$$\int\frac{1}{x^4+1}dx=\int\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}-\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}\right)dx$$
$$+\frac{1}{4}\int\left(\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right)dx \quad ......(3)$$

The formulas are $\int\frac{f'(x)}{f(x)}dx=\ln f(x)+C$ and $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)+C$.

$$=\frac{\sqrt{2}}{8}\left(\ln\left(x^2+\sqrt{2}x+1\right)-\ln\left(x^2-\sqrt{2}x+1\right)\right)$$
$$+\frac{\sqrt{2}}{4}\left(\tan^{-1}\left(\sqrt{2}x+1\right)+\tan^{-1}\left(\sqrt{2}x-1\right)\right)+C$$
$$=\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\left(\tan^{-1}\left(\sqrt{2}x+1\right)+\tan^{-1}\left(\sqrt{2}x-1\right)\right)+C$$

Hence, the answer is $\boxed{\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\left(\tan^{-1}\left(\sqrt{2}x+1\right)+\tan^{-1}\left(\sqrt{2}x-1\right)\right)+C}$.

Answer 69E.

(a)

Consider the function:

$$f(x)=\frac{4x^3-27x^2+5x-32}{30x^5-13x^4+50x^3-286x^2-299x-70}.$$

Use Maple, to find the partial fraction decomposition.

Maple Input:

```
f:=(4x^3-27x^2+5x-32)/(30x^5-13x^4+50x^3-286x^2-299x-70)
convert(f,parfrac,x);
```

Maple Output:

$$> f := \frac{(4x^3-27x^2+5x-32)}{30x^5-13x^4+50x^3-286x^2-299x-70}$$

$$f := \frac{4x^3-27x^2+5x-32}{30x^5-13x^4+50x^3-286x^2-299x-70}$$

> *convert*(*f*, *parfrac*, *x*);

$$-\frac{668}{323(2x+1)}+\frac{1}{260015}\frac{22098x+48935}{x^2+x+5}+\frac{24110}{4879(5x+2)}-\frac{9438}{80155(3x-7)}$$

(b)

Evaluate the integral $\int f(x)\,dx$.

$$\int \frac{4x^3-27x^2+5x-32}{30x^5-13x^4+50x^3-286x^2-299x-70}$$

$$=\int -\frac{668}{323(2x+1)}dx+\int \frac{1}{260015}\frac{22098x+48935}{x^2+x+5}dx$$

$$+\int \frac{24110}{4879(5x+2)}dx-\int \frac{9438}{80155(3x-7)}dx$$

$$=-\frac{668}{323}\left(\frac{1}{2}\right)\ln|2x+1|+\int \frac{1}{260015}\frac{22098x+48935}{x^2+x+5}dx$$

$$+\frac{24110}{4879}\left(\frac{1}{5}\right)\ln|5x+2|-\frac{9438}{80155}\left(\frac{1}{3}\right)\ln|3x-7| \text{ ...... (1)}$$

Evaluate the second term of the integral.

$$\int \frac{1}{260015}\frac{22098x+48935}{x^2+x+5}dx$$

$$=\frac{1}{260015}\int \frac{22098x}{x^2+x+5}dx+\frac{48935}{260015}\int \frac{1}{x^2+x+5}dx$$

$$=\frac{11049}{260015}\int \frac{2x+1-1}{x^2+x+5}dx+\frac{48935}{260015}\int \frac{1}{x^2+x+5}dx$$

$$=\frac{11049}{260015}\int \frac{2x+1}{x^2+x+5}dx-\frac{11049}{260015}\int \frac{1}{x^2+x+5}$$

$$+\frac{48935}{260015}\int \frac{1}{x^2+x+5}dx$$

$$=\frac{11049}{260015}\int \frac{2x+1}{x^2+x+5}dx-\frac{37886}{260015}\int \frac{1}{x^2+x+5}dx$$

$$=\frac{11049}{260015}\ln|x^2+x+5|-\frac{37886}{260015}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{19}{4}\right)}dx$$

$$=\frac{11049}{260015}\ln|x^2+x+5|-\frac{37886}{260015}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{19}}{2}\right)^2}dx$$

$$=\frac{11049}{260015}\ln|x^2+x+5|-\left(\frac{37886}{260015}\right)\frac{2}{\sqrt{19}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{19}}{2}}\right)$$

$$=\frac{11049}{260015}\ln|x^2+x+5|-\left(\frac{37886}{260015}\right)\frac{2}{\sqrt{19}}\tan^{-1}\left(\frac{2x+1}{\sqrt{19}}\right)$$

$$=\frac{11049}{260015}\ln|x^2+x+5|-\left(\frac{75772}{260015\sqrt{19}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{19}}\right) \text{ ...... (2)}$$

Substitute the (2) in (1).

$$\int \frac{4x^3-27x^2+5x-32}{30x^5-13x^4+50x^3-286x^2-299x-70}$$

$$= -\frac{668}{323}\left(\frac{1}{2}\right)\ln|2x+1| + \int \frac{1}{260015}\frac{22098x+48935}{x^2+x+5}dx$$

$$+\frac{24110}{4879}\left(\frac{1}{5}\right)\ln|5x+2| - \frac{9438}{80155}\left(\frac{1}{3}\right)\ln|3x-7|$$

$$= -\frac{334}{323}\ln|2x+1| + \frac{11049}{260015}\ln|x^2+x+5| - \left(\frac{75772}{260015\sqrt{19}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{19}}\right)$$

$$+\frac{4822}{4879}\ln|5x+2| - \frac{3146}{80155}\ln|3x-7| + C$$

Therefore, the integration of $f(x)$ is:

$$\boxed{-\frac{334}{323}\ln|2x+1| + \frac{11049}{260015}\ln|x^2+x+5| - \left(\frac{75772}{260015\sqrt{19}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{19}}\right) + \frac{4822}{4879}\ln|5x+2| - \frac{3146}{80155}\ln|3x-7| + C}$$

Answer 70E.

Using a computer algebra system we need to find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5-7x^3-13x^2+8}{100x^6-80x^5+116x^4-80x^3+41x^2-20x+4}$$

Using the maple command we can find the partial fraction as shown below:

```
> f:=(12*x^5-7*x^3-13*x^2+8)/(100*x^6-80*x^5+116*x^4-80*x^3+41*x^2-20*x+4);
```

$$f := \frac{12x^5-7x^3-13x^2+8}{100x^6-80x^5+116x^4-80x^3+41x^2-20x+4}$$

```
> convert(f,parfrac,x);
```

$$-\frac{59096}{19965(5x-2)} + \frac{5828}{1815(5x-2)^2} + \frac{1632+5686x}{3993(2x^2+1)} + \frac{-251+313x}{363(2x^2+1)^2}$$

```
> convert(f,parfrac);
```

$$-\frac{59096}{19965(5x-2)} + \frac{5828}{1815(5x-2)^2} + \frac{1632+5686x}{3993(2x^2+1)} + \frac{-251+313x}{363(2x^2+1)^2}$$

Using part (a) we need to find $\int f(x)dx$ and graph f and its indefinite integral on the same screen.

We can find $\int f(x)dx$ as

$$I = \int f(x)dx$$

$$= \int \frac{12x^5-7x^3-13x^2+8}{100x^6-80x^5+116x^4-80x^3+41x^2-20x+4}dx$$

$$= \int \frac{-59096}{19965(5x-2)}dx + \int \frac{5828}{1815(5x-2)^2}dx + \int \frac{1632+5686x}{3993(2x^2+1)}dx + \int \frac{-251+313x}{363(2x^2+1)^2}dx$$

$$= -\frac{0.6422}{5x-2} - 0.59199\ln(5x-2)$$

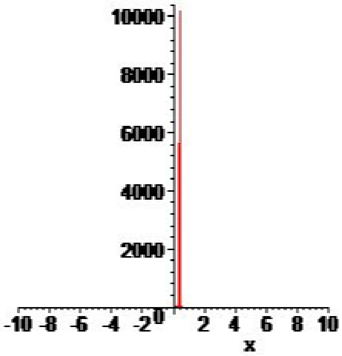
$$+\frac{0.000344(-1004x-626)}{2x^2+1} + 0.04453\tan^{-1}(1.4142x)$$

$$+0.355999\ln(2x^2+1)$$

We can graph *f* as shown below:

Using the maple command we have

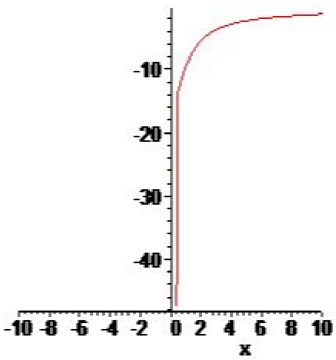
```
> with(plots):
> a1:=plot(-.6422038567/(5.*x-2.)-.5919959930*ln(5.*x-2.)+.3443526171*exp(-3)*(-1004.*x-626.)/(2.*x^2+1.)+.4453727909e-1*arctan(1.414213562*x)+.3559979965*ln(2.*x^2+1.), x=-10...10):
> a2:=plot((12*x^5-7*x^3-13*x^2+8)/(100*x^6-80*x^5+116*x^4-80*x^3+41*x^2-20*x+4),x=-10..10):
> display(a1,a2);
```



Using a graph we need to discover the main features of the graph of $\int f(x)dx$.

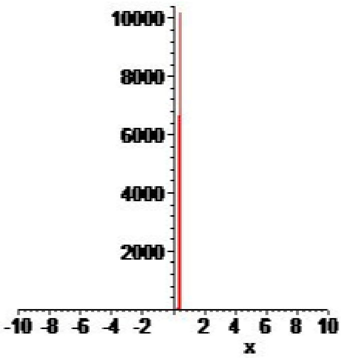
Using the maple command the integral of $f(x)$ we have

```
> with(plots):
> a1:=plot(-.6422038567/(5.*x-2.)-.5919959930*ln(5.*x-2.)+.3443526171*exp(-3)*(-1004.*x-626.)/(2.*x^2+1.)+.4453727909e-1*arctan(1.414213562*x)+.3559979965*ln(2.*x^2+1.), x=-10...10);
```



The graph of $f(x)$ can be drawn from the integral of $f(x)$ as:

```
> with(plots):
> a1:=plot((12*x^5-7*x^3-13*x^2+8)/(100*x^6-80*x^5+116*x^4-80*x^3+41*x^2-20*x+4),x=-10..10):
> display(a1);
```



Answer 71E.

Suppose that F, G and Q are polynomials and $\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$ for all x except when

$Q(x) = 0$.

Prove that $F(x) = G(x)$ for all x.

Assume $Q(x) \neq 0$

The polynomials F, G, Q are all continuous, cross multiply equation,

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$
$$F(x)Q(x) = G(x)Q(x)$$

For all other values of x, divide by $Q(x)$

$$F(x)Q(x) = G(x)Q(x)$$
$$\frac{F(x)Q(x)}{Q(x)} = \frac{G(x)Q(x)}{Q(x)}$$
$$F(x) = G(x)$$

Hence, we proved i.e. $\boxed{F(x) = G(x)}$ for all x.

Let a is a value of x such that $Q(a) = 0$ then $Q(x) \neq 0$ for all x close to a.

So,

$$\begin{aligned} F(a) &= \lim_{x \to a} F(x) \text{ (From continuity of } F) \\ &= \lim_{x \to a} G(x) \text{ (whenever } Q(x) \neq 0) \\ &= G(a) \quad \text{(From continuity of } G) \end{aligned}$$

Hence, we proved i.e. $\boxed{F(x) = G(x)}$ for all x.

Answer 72E.

If the function $f(x) = ax^2 + bx + c$ such that $f(0) = 1$ and $\int \frac{f(x)}{x^2(x+1)^3} dx$ is a rational

function.

Find the value of $f'(0)$

Substitute the value $f(0) = 1$ in quadratic function,

$$f(x) = ax^2 + bx + c$$
$$f(0) = a(0)^2 + b(0) + c$$
$$1 = c$$

Hence, the quadratic function is $f(x) = ax^2 + bx + 1$.

To find the value of $f'(0)$, differentiate the quadratic function and then substitute $x = 0$.

Apply first derivative to the quadratic function,

$$f(x) = ax^2 + bx + 1$$
$$f'(x) = 2ax + b$$

Substitute the value $x = 0$ in above function,

$$f'(0) = b \quad \text{......(1)}$$

To find the value of b, rewrite the rational function $\frac{f(x)}{x^2(x+1)^3}$ as partial fractions,

$$\frac{f(x)}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$
$$\frac{ax^2+bx+1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

The integral $\int \frac{f(x)}{x^2(x+1)^3}dx$ is a rational function, which means that there can be no logarithms in the answer i.e. the integral of the functions $\frac{A}{x}$ and $\frac{C}{x+1}$ are logarithms. But the answer cannot have logarithms so we need take $A=0$ and $C=0$.

The partial fractions will becomes,

$$\frac{ax^2+bx+1}{x^2(x+1)^3} = \frac{B}{x^2} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

Cancel the common denominators,

$$\begin{aligned} ax^2+bx+1 &= +B(x+1)^3 + +Dx^2(x+1) + Ex^2 \\ &= B(x^3+3x^2+3x+1) + Dx^2(x+1) + Ex^2 \\ &= x^3(B+D) + x^2(3B+D+E) + x(3B) + (B) \end{aligned}$$

Compare x^3 coefficients on both sides,

$$B+D=0$$

Compare x^2 coefficients on both sides,

$$3B+D+E=a$$

Compare the constants on both sides,

$$B=1$$

Compare x coefficients on both sides,

$$3B=b$$
$$3=b\ (\text{Since}\ B=1)$$

From the equation – (1),

$$\begin{aligned} f'(0) &= b \\ &= 3 \end{aligned}$$

Hence, the answer is $\boxed{f'(0)=3}$.

Answer 73E.

Given $f(x)=\dfrac{1}{x^n(x-a)}, a \neq 0$

$$\frac{1}{x^n(x-a)}=\frac{A_1}{x-a}+\frac{A_2}{x}+\frac{A_3}{x^2}+\cdots+\frac{A_{n+1}}{x^n}$$

$$\Rightarrow \quad A_1x^n+A_2x^{n-1}(x-a)+A_3x^{n-2}(x-a)+\cdots+A_{n-1}(x-a)=1$$

$$x=a \quad \Rightarrow \quad A_1a^n=1 \quad \Rightarrow \quad A_1=\frac{1}{a^n}$$

Equate the coefficients of x^n on both sides

$$A_1+A_2=0 \quad \Rightarrow \quad A_2=-\frac{1}{a^n}$$

Equate the coefficients of x^{n-1} on both sides

$$-A_2a+A_3=0 \Rightarrow \quad A_3=aA_2$$

$$=-\frac{a}{a^n}$$

$$=\frac{-1}{a^{n-1}}$$

Continuing like this we get $A_4=-\dfrac{1}{a^{n-2}},\cdots$

$$A_{n+1}=-\frac{1}{a}$$

Therefore $f(x)=\dfrac{1}{x^n(x-a)}$

$$=\frac{1}{a^n(x-a)}-\frac{1}{a^nx}-\frac{1}{a^{n-1}x^2}-\cdots-\frac{1}{ax^n}$$