Exercise 11.10

Q1E

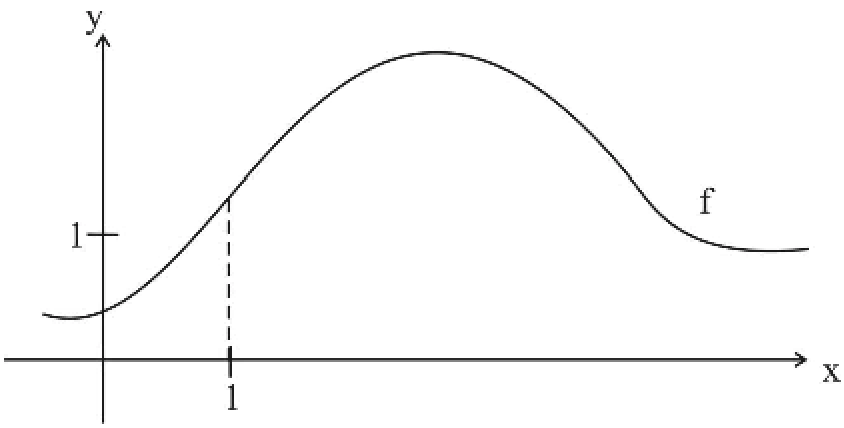
It $f(x)=\sum_{n=0} b_n(x-5)^n$ for all x, then

Then $a=5,\quad b_n=\frac{f^n(a)}{n!}=\frac{f^n(5)}{n!}$

And therefore $\boxed{b_8=\frac{f^{(8)}(5)}{8!}}$

Q2E

(A)



The given series is

$$1.6-0.8(x-1)+0.4(x-1)^2-0.1(x-1)^3+\ldots\ldots----(1)$$

Also if f has a power series expansion centered at a then,

$$f(x)=f(a)+\frac{f'(a)(x-a)}{1!}+\frac{f''(a)(x-a)^2}{2!}+\frac{f'''(a)(x-a)^3}{3!}+----$$

Therefore,

$$f(x)=f(1)+\frac{f'(1)(x-1)}{1!}+\frac{f''(1)(x-1)^2}{2!}+\frac{f'''(1)(x-1)^3}{3!}+----(2)$$

Comparing (1) and (2) we get

$$f'(1)=-0.8$$

$\Rightarrow \quad f'(1)<0 \quad$ *i.e.* $\quad f$ is decreasing at $x=1$.

Now from the graph it is clear that at x = 1. The graph of the function f is increasing i. e. $f'(1)>0$

Therefore,

The given series can not be the Taylor series of f centered at $x=1$.

(B) The given series is,

$$2.8+0.5(x-2)+1.5(x-2)^2-0.1(x-2)^3+\ldots\ldots(1)$$

The Taylor series for a function f centered at

$x=2$ is

$$f(x)=f(2)+\frac{f'(2)(x-2)}{1!}+\frac{f''(2)(x-2)^2}{2!}+\frac{f'''(2)(x-2)^3}{3!}+\ldots\ldots(2)$$

Comparing coefficients of $(x-2)^2$ form (1) and (2)

We get

$$\frac{f''(2)}{2!}=1.5$$

$\Rightarrow \quad f''(2)=1.5\times 2=3>0$

i.e. $\quad f''(2)$ is positive telling us that $f(x)$ is concave upward near $x=2$.

Now from the graph it is clear that the graph of the function $f(x)$ is concave downwards near $x=2$

i. e. $\quad f'(2)<0$

Therefore, the given series is not a Taylor series for f centered at 2.

Q3E

Consider $f^{(n)}(0)=(n+1)!$ for $n=0,1,2,\ldots$

Recollect the Maclaurin series for the function *f* states that

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots$$

To find the Maclaurin series for the function *f*.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$=\sum_{n=0}^{\infty}\frac{(n+1)!}{n!}x^n$ Substitute $f^{(n)}(0)=(n+1)!$

$=\sum_{n=0}^{\infty}\frac{(n+1)n!}{n!}x^n$ Since $(n+1)!=(n+1)n!$

$=\sum_{n=0}^{\infty}(n+1)x^n$ Simplify.

So the Maclaurin series for the function *f* is

$$f(x)=\boxed{\sum_{n=0}^{\infty}(n+1)x^n}.$$

The ratio test:

The series $\sum_{n=0}^{\infty} a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$

Recollect the radius R of convergence for the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is follows: If $|x-a|<R$, then the series converges If $|x-a|>R$, then the series diverges

To find the radius of convergence for the series $\sum_{n=0}^{\infty}(n+1)x^n$:

Let $a_n=(n+1)x^n$, then $a_{n+1}=(n+2)x^{n+1}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$
$$=|x|\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)$$

$=|x|\lim_{n\to\infty}\left(\frac{1+2/n}{1+1/n}\right)$ Divide *n* by the numerator and denominator.

$=|x|$ Since $\lim_{n\to\infty}(1/n)=0$

Since $|x|<1$, by the ratio test, the series $\sum_{n=0}^{\infty}(n+1)x^n$ converges.

Therefore the radius of convergence is $\boxed{R=1}$.

To find the radius of convergence for the series $\sum_{n=0}^{\infty}(n+1)x^n$:

Let $a_n=(n+1)x^n$, then $a_{n+1}=(n+2)x^{n+1}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$
$$=|x|\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)$$

$=|x|\lim_{n\to\infty}\left(\frac{1+2/n}{1+1/n}\right)$ Divide *n* by the numerator and denominator.

$=|x|$ Since $\lim_{n\to\infty}(1/n)=0$

Since $|x|<1$, by the ratio test, the series $\sum_{n=0}^{\infty}(n+1)x^n$ converges.

Therefore the radius of convergence is $\boxed{R=1}$.

Q4E

Ratio test: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, the series diverges, and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, ratio test is inconclusive.

Taylor's series of the function f centered at a is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Given function is $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$

So, the corresponding Taylor series centered at 4 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n(n+1)\cdot n!}(x-4)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$$

To find the radius of convergence, we consider $a_n = \frac{(-1)^n}{3^n(n+1)}(x-4)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-4)^{n+1}}{3^{n+1}(n+2)}}{\frac{(x-4)^n}{3^n(n+1)}}\right|$$

$$= \frac{1}{3}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)|x-4|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\left(\frac{1+0}{1+0}\right)|x-4|$$

By Ratio test, the series converges when $\frac{|x-4|}{3} < 1$,

$$\Rightarrow -3 < x-4 < 3$$

$$\Rightarrow 1 < x < 7$$

So, radius of convergence is 3 and the interval of convergence is (1, 7).

Q5E

Consider the function $f(x)=(1-x)^{-2}$.

To find the Maclaurin series for $f(x)$, use the power series expansion of $f(x)$.

Power series explanation of a function $f(x)$:

If f has a power series representation at *a*, then it must be of the following form.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}(x-a)^n \qquad |x-a|<R.$$

Or

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2++\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Therefore the power series for f at 0 that is the Maclaurin series is

$$\begin{aligned}f(x)&=\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}(x-0)^n\\ &=f(0)+\frac{f'(0)}{1!}(x-0)+\frac{f''(0)}{2!}(x-0)^2++\frac{f'''(0)}{3!}(x-0)^3+\cdots\\ &=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots\end{aligned}$$

So at $a=0$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots \qquad |x|<R$$

...... (1)

If $f(x)=(1-x)^{-2}$, then at $x=0$, get

$$\begin{aligned}f(0)&=(1-0)^{-2}\\ &=1\end{aligned}$$

Find the first derivative of $f(x)$: Differentiate $f(x)$ with respect to x, get

$f'(x)=-2(1-x)^{-3}(-1)$ (Since if $f(x)=x^n$ then $f'(x)=nx^{n-1}$)

$=2(1-x)^{-3}$

At $x=0$, get

$$\begin{aligned}f'(0)&=2(1-0)^{-3}\\ &=2\end{aligned}$$

Find the second derivative of $f(x)$: Differentiate $f'(x)$ again with respect to x, get

$$f''(x) = 2(-3)(1-x)^{-4}(-1)$$
$$= 6(1-x)^{-4}$$

At $x = 0$, get

$$f''(0) = 6(1-0)^{-4}$$
$$= 6$$

Find the third derivative of $f(x)$: Differentiate $f''(x)$ again with respect to x, get

$$f'''(x) = 6(-4)(1-x)^{-5}(-1)$$
$$= 24(1-x)^{-5}$$

At $x = 0$, get

$$f'''(0) = 24(1-0)^{-5}$$
$$= 24$$

And this pattern repeats indefinitely. Therefore the Maclaurin series of $f(x) = (1-x)^{-2}$ is

$$f(x) = (1-x)^{-2}$$
$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$
$$= 1 + \frac{2}{1!}x + \frac{6}{2!}x^2 + \frac{24}{3!}x^3 + \cdots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$= \sum_{n=0}^{\infty}(n+1)x^n$$

To find the radius of convergence, use the Ratio Test with $a_n = (n+1)x^n$.

Ratio Test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Let $a_n = (n+1)x^n$. Then

$$a_{n+1} = (n+1+1)x^{n+1}$$
$$= (n+2)x^{n+1}$$

And

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)x^{n+1}}{(n+1)x^n}$$
$$= \frac{(n+2)x^n \cdot x}{(n+1)x^n}$$
$$= \frac{n\left(1+\frac{2}{n}\right)x}{n\left(1+\frac{1}{n}\right)}$$
$$= \frac{\left(1+\frac{2}{n}\right)x}{\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{2}{n}\right)x}{\left(1+\frac{1}{n}\right)}\right|$$
$$= \lim_{\frac{1}{n}\to 0}\left|\frac{\left(1+\frac{2}{n}\right)x}{\left(1+\frac{1}{n}\right)}\right|$$
$$= \left|\frac{(1+0)x}{(1+0)}\right|$$
$$= |x|$$

So by Ratio test this series is converges when $|x| < 1$, so the radius of convergence is $R = 1$.

Hence the Maclaurin series of $f(x) = (1-x)^{-2}$ is $f(x) = \boxed{1+2x+3x^2+4x^3+\cdots \quad |x|<1}$.

Q6E

We have $f(x) = \ln(1+x)$ $\qquad f(0) = \ln(1) = 0$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \qquad f'(0) = 1 = (-1)^0\, 0!$$
$$f''(x) = (-1)(1+x)^{-2} \qquad f''(0) = (-1) = (-1)^1\, 1!$$
$$f'''(x) = (-1)(-2)(1+x)^{-3} \qquad f'''(0) = (-1)(-2) = (-1)^2 .1.2. = (-1)^2\, 2!$$
$$f^{(iv)}(x) = (-1)(-2)(-3)(1+x)^{-4} \qquad f^{(iv)}(0) = (-1)(-2)(-3) = (-1)^3\, 1.2.3. = (-1)^3\, 3!$$

…………………………………………….

…………………………………………….

The Maclaurin's series is as follows

$$f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^{(4)}(0)x^4}{4!}+\ldots\ldots$$

$$=(-1)^0\frac{0!x}{1!}+\frac{(-1)^1 1!x^2}{2!}+\frac{(-1)^2 2!x^3}{3!}+\frac{(-1)^3 3!x^4}{4!}+\ldots\ldots$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!x^n}{n!}$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!x^n}{n(n-1)!}$$

$$=\boxed{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}}$$

Now

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right|$$

$$=\lim_{n\to\infty}\frac{|x|}{1+1/n}=|x|$$

Thus by the Ratio test the given series converges for $|x|<1$ and radius for convergence is $\boxed{R=1.}$

Q7E

Consider the function

$f(x)=\sin\pi x$.

Arrange the computation in two columns as follows:

$$\begin{array}{ll} f(x)=\sin\pi x & f(0)=0 \\ f'(x)=\pi\cos\pi x & f'(0)=-\pi \\ f''(x)=-\pi^2\sin\pi x & f''(0)=0 \\ f'''(x)=-\pi^3\cos\pi x & f'''(0)=\pi^3 \\ f^{iv}(x)=\pi^4\sin\pi x & f^{iv}(0)=0 \end{array}$$

Note that the fourth derivative takes one back to the start point with a multiple of π^4, so these values repeat in cycle of four as $0,\pi,0,-\pi^3,0,\pi^5,0,-\pi^7,\ldots$

The Maclaurin Series of the function $f(x)$ is as follows:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

Substitute the values to the Maclaurin Series.

$$\begin{aligned} f(x) &= f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots \\ &= 0+\frac{-\pi}{1!}x+\frac{0}{2!}x^2+\frac{\pi^3}{3!}x^3+\frac{0}{4!}x^4+\frac{-\pi^5}{5!}x^5+\frac{0}{6!}x^6+\frac{\pi^7}{7!}x^7+\cdots \\ &= \frac{-\pi}{1!}x+\frac{\pi^3}{3!}x^3+\frac{-\pi^5}{5!}x^5+\frac{\pi^7}{7!}x^7+\cdots \\ &= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}x^{2n+1}} \end{aligned}$$

To find the radius of convergence, solve as follows:

Let $a_n=\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}x^{2n+1}$, then proceed as follows:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}\pi^{2n+3}}{(2n+3)!}x^{2n+3}}{\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}x^{2n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\pi^{2n+3}}{(2n+3)!}x^{2n+3}\cdot\frac{(2n+1)!}{(-1)^n\pi^{2n+1}x^{2n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1).\pi^2}{(2n+3)(2n+2)}x^2\right| \end{aligned}$$

$$\begin{aligned} &= \lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\pi^2x^2\right| \\ &= \left|(\pi x)^2\right|\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} \\ &\simeq 0 \\ &< 1 \end{aligned}$$

The Ratio Test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L(<1)$, then the series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L(>1)$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$, then the series $\sum_{n=1}^{\infty}a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, then the Ratio test is inconclusive.

Therefore, by the Ratio Test, the series converges for all x and the radius of convergence is $\boxed{R=\infty}$.

Q8E

To find the **Maclaurin series** of a function, consider the function

$$f(x) = e^{-2x}$$

Maclaurin series of a function $f(x)$ is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$
$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

So, it is need to compute $f(0), f'(0), f''(0), f'''(0), \ldots$

As $f(x) = e^{-2x}$,

$$f(0) = e^{-2(0)}$$
$$= e^0$$
$$= 1$$

So, $f(0) = 1$

As $f(x) = e^{-2x}$, on differentiating it

$$f'(x) = -2e^{-2x}$$

Now, $f'(0) = -2e^{-2(0)}$

$$= -2(e^0)$$
$$= -2(1)$$
$$= -2$$

So, $f'(0) = -2$

As $f'(x) = -2e^{-2x}$, on differentiating it

$$f''(x) = 4e^{-2x}$$

Now, $f''(0) = 4e^{-2(0)}$

$$= 4(e^0)$$
$$= 4(1)$$
$$= 4$$

So, $f''(0) = 4$

As $f''(x)=4e^{-2x}$, on differentiating it

$f'''(x)=-8e^{-2x}$

Now, $f'''(0)=-8e^{-2(0)}$

$=-8(e^{0})$

$=-8(1)$

$=-8$

So, $f'''(0)=-8$

As $f'''(x)=-8e^{-2x}$, on differentiating it

$f^{iv}(x)=16e^{-2x}$

Now, $f^{iv}(0)=16e^{-2(0)}$

$=16(e^{0})$

$=16(1)$

$=16$

So, $f^{iv}(0)=16$ and this pattern repeats indefinitely.

Therefore, Maclaurin series of the function $f(x)=e^{-2x}$ is

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^3+...$$

$$=1+\frac{(-2)}{1!}x+\frac{4}{2!}x^2+\frac{(-8)}{3!}x^3+\frac{16}{4!}x^4+....$$

$$=1-2x+\frac{4}{2}x^2+\frac{(-8)}{6}x^3+\frac{16}{24}x^4+....$$

$$=1-2x+\frac{4x^2}{2}-\frac{8x^3}{6}+\frac{16x^4}{24}+...$$

The sum of the series

$$1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6} + \frac{16x^4}{24} + \ldots$$

can be written as

$$= \frac{(-2)^0}{0!}x^0 + \frac{(-2)^1}{1!}x^1 + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n$$

Hence, $e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n$

To find the associated radius of convergence of $f(x) = e^{-2x}$,

consider its summation notation, that is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n$$

Recall that, for the power series $\sum a_n x^n$, define **radius of convergence** R as

$$R = \frac{1}{\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

If $R = 0$, the series is nowhere convergent and if $R = \infty$, the series is everywhere convergent.

For the series, $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n$

Let $a_n = \frac{(-2)^n x^n}{n!}$, then $a_{n+1} = \frac{(-2)^{n+1} x^{n+1}}{(n+1)!}$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{2^{n+1}\left|x^{n+1}\right|}{(n+1)!} \cdot \frac{n!}{2^n \left|x^n\right|} \\
&= \lim_{n\to\infty} \frac{2|x|}{n+1} \\
&= \frac{1}{\infty} \qquad \text{Because } as\ n \to \infty,\ n+1 \to \infty \\
&= 0
\end{aligned}$$

As $R=\dfrac{1}{\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|}$ and $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=0$, it follows that

$$\begin{aligned}
R&=\frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}\\
&=\frac{1}{0}\\
&=\infty
\end{aligned}$$

Hence, the series has radius of convergence $\boxed{\infty}$,that is the series is everywhere convergent.

Q9E

To find the **Maclaurin series** of a function, consider the function

$f(x)=2^x$

Maclaurin series of a function $f(x)$ is defined as

$$\begin{aligned}
f(x)&=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n\\
&=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots
\end{aligned}$$

So, it is need to compute $f(0),f'(0),f''(0),f'''(0),\dots$

As $f(x)=2^x$,

$$\begin{aligned}
f(0)&=2^0\\
&=1
\end{aligned}$$

So, $f(0)=1$

As $f'(x) = 2^x(\ln 2)$, on differentiating it

$f''(x) = 2^x(\ln 2)^2$

Now, $f''(x) = 2^0(\ln 2)^2$

$= 1(\ln 2)^2$

$= (\ln 2)^2$

So, $f''(0) = (\ln 2)^2$

As $f''(x) = 2^x(\ln 2)^2$, on differentiating it

$f'''(x) = 2^x(\ln 2)^3$

Now, $f'''(0) = 2^0(\ln 2)^3$

$= 1(\ln 2)^3$

$= (\ln 2)^3$

So, $f'''(0) = (\ln 2)^3$

As $f'''(x) = 2^x(\ln 2)^3$, on differentiating it

$f^{iv}(x) = 2^x(\ln 2)^4$

Now, $f^{iv}(0) = 2^0(\ln 2)^4$

$= 1(\ln 2)^4$

$= (\ln 2)^4$

So, $f^{iv}(0) = (\ln 2)^4$ and this pattern repeats indefinitely.

Therefore, Maclaurin series of the function $f(x)=2^x$ is

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^3+\ldots$$
$$=1+\frac{(\ln 2)}{1!}x+\frac{(\ln 2)^2}{2!}x^2+\frac{(\ln 2)^3}{3!}x^3+\frac{(\ln 2)^4}{4!}x^4+\ldots.$$

The sum of the series

$$1+\frac{(\ln 2)}{1!}x+\frac{(\ln 2)^2}{2!}x^2+\frac{(\ln 2)^3}{3!}x^3+\frac{(\ln 2)^4}{4!}x^4+\ldots.$$

can be written as

$$=\frac{(\ln 2)^0}{0!}x^0+\frac{(\ln 2)^1}{1!}x^1+\frac{(\ln 2)^2}{2!}x^2+\frac{(\ln 2)^3}{3!}x^3+\frac{(\ln 2)^4}{4!}x^4+\ldots.$$
$$=\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!}x^n$$

Hence, $2^x=\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!}x^n$

To find the associated **radius of convergence** of $f(x)=2^x$,

consider its summation notation, that is

$$\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!}x^n$$

Recall that, for the power series $\sum a_n x^n$, define radius of convergence R as

$$R=\frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

If $R=0$, the series is nowhere convergent and if $R=\infty$, the series is everywhere convergent.

For the series, $\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!}x^n$

Let $a_n=\frac{(\ln 2)^n}{n!}x^n$, then $a_{n+1}=\frac{(\ln 2)^{n+1}}{(n+1)!}x^{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(\ln 2)^{n+1}\left|x^{n+1}\right|}{(n+1)!}\cdot\frac{n!}{(\ln 2)^n\left|x^n\right|}$$
$$=\lim_{n\to\infty}\frac{(\ln 2)|x|}{n+1}$$
$$=\frac{1}{\infty}\qquad \text{Because } as\ n\to\infty,\ n+1\to\infty$$
$$=0$$

Q10E

The Ratio test:

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent,

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, the series divergent, and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, ratio test is inconclusive

Maclaurin series: $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}}{n!}x^n + \ldots$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Consider the function $f(x) = x\cos x$

Need to find the required values

$f(0) = 0$

Take

$f(x) = x\cos x$

Differentiation with respect to x on both sides

$f'(x) = -x\sin x + \cos x$ Since $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

$f'(0) = -(0)\sin(0) + \cos(0)$

$f'(0) = 1$

Take

$f'(x) = -x\sin x + \cos x$

Again differentiation with respect to x on both sides

$f''(x) = -x\cos x - 2\sin x$ Since $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

$f''(0) = -(0)\cos(0) - 2\sin(0)$

$f''(0) = 0$

Take $f''(x) = -x\cos x - 2\sin x$

$f''(x) = -f(x) - 2\sin x$ Since $f(x) = x\cos x$

Again differentiation with respect to x on both sides

$f'''(x) = -f'(x) - 2\cos x$

$f'''(0) = -f'(0) - 2\cos(0)$

$f'''(0) = -1 - 2(1)$ Since $f'(0) = 1$

$f'''(0) = -3$

Take

$f'''(x) = -f'(x) - 2\cos x$

Again differentiation with respect to x on both sides

$f^{(4)}(x) = -f''(x) + 2\sin x$

$f^{(4)}(0) = -f''(0) + 2\sin(0)$

$f^{(4)}(0) = -0 + 2(0)$ Since $f''(0) = 0$

$f^{(4)}(0) = 0$

Take

$f^{(4)}(x) = -f''(x) + 2\sin x$

Again differentiation with respect to x on both sides

$f^{(5)}(x) = -f'''(x) + 2\cos x$

$f^{(5)}(0) = -f'''(0) + 2\cos(0)$

$f^{(5)}(0) = -(-3) + 2(1)$ Since $f'''(0) = -3$

$f^{(5)}(0) = 5$

Since the derivatives repeat in a cycle of five, write the Maclaurin series as

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

$$=0+x+\frac{0}{2!}x^2-\frac{3}{3!}x^3+\frac{0}{4!}x^4+\frac{5}{5!}x^5-\ldots$$

$$=x-\frac{3}{3!}x^3+\frac{5}{5!}x^5-\ldots$$

$$=x-\frac{3}{3\cdot 2!}x^3+\frac{5}{5\cdot 4!}x^5-\ldots$$

$$=x-\frac{1}{2!}x^3+\frac{1}{4!}x^5-\ldots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}$$

Thus, $x\cos x=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}}$.

Need to find the radius of convergence

Let $a_n=\frac{x^{2n+1}}{(2n)!}$

Keeping ratio test in view,

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\left(\frac{x^{2(n+1)+1}}{(2(n+1))!}\right)}{\left(\frac{x^{2n+1}}{(2n)!}\right)}\right|$$ Replace n with $n+1$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{2n+3}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n+1}}\right|$$ Multiply

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{(2n+1)+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n+1}}\right|$$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{2n+1}\cdot x^2}{(2n+2)(2n+1)(2n)!}\cdot\frac{(2n)!}{x^{2n+1}}\right|$$

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{x^2}{(2n+2)(2n+1)}$$ Cancel like terms

Continuation to the above steps,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{n\left(2+\frac{2}{n}\right)n\left(2+\frac{1}{n}\right)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{n(2+0)n(2+0)}$$ As $n \to \infty$

Note that as $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to 0$

Then, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ for all n

Therefore,

By the Ratio test, the series converges for all x and the radius of converges is $R = \boxed{\infty}$.

Q11E

.

We have

$$f(x) = \sinh x \qquad f(0) = 0$$
$$f'(x) = \cosh x \qquad f'(0) = 1$$
$$f''(x) = \sinh x \qquad f''(0) = 0$$
$$f'''(x) = \cosh x \qquad f'''(0) = 1$$

....

....

The Maclaurin's series is as follows

$$f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(iv)}(0)x^4}{4!} + \ldots..$$
$$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots..$$
$$= \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$$

Now $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\frac{(2n+1)!}{x^{2n+1}}\right|$$
$$= \lim_{n\to\infty}\frac{|x^2|}{(2n+3)(2n+2)} = 0 < 1 \quad \text{for all values of x}$$

Thus by the Ratio test, the given series converges for all values of x. Hence the radius of convergence, $\boxed{R = \infty}$

Q12E

We have
$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= \sinh x & f'(0) &= 0 \\ f''(x) &= \cosh x & f''(0) &= 1 \\ f'''(x) &= \sinh x & f'''(0) &= 0 \\ f^{(iv)}(x) &= \cosh x & f^{(iv)}(0) &= 1 \\ f^{(v)}(x) &= \sinh x & f^{(v)}(0) &= 0 \end{aligned}$$
....

....

Then Maclaurin's series is as follows:

$$f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^{(iv)}(0)x^4}{4!}+........$$
$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+......$$
$$=\boxed{\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}}$$

Now $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\frac{(2n)!}{x^{2n}}\right|$
$$=\lim_{n\to\infty}\frac{|x^2|}{(2n+2)(2n+1)}=0<1 \quad \text{for all } x$$

Thus by the Ratio Test, The gives series converges for all values of x. Hence the radius convergence is $\boxed{R=\infty}$.

Q13E

Consider the function,

$$f(x)=x^4-3x^2+1.$$

The objective is to find the Taylor series for $f(x)$ at center $a=1$ and also find the radius of convergence.

Recall that Taylor series is,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+... \quad .... (1)$$

Using (1) Taylor series become

$$f(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^4(1)}{4!}(x-1)^4+\dots$$

Arrange our computation in two columns as follows:

$$\begin{array}{ll} f(x)=x^4-3x^2+1 & f(1)=-1 \\ f'(x)=4x^3-6x & f'(1)=-2 \\ f''(x)=12x^2-6 & f''(1)=6 \\ f'''(x)=24x & f'''(1)=24 \\ f^{(4)}(x)=24 & f^{(4)}(1)=24 \end{array}$$

So, Taylor series representation $f(x)=x^4-3x^2+1$ centered at $a=1$ is,

$$\begin{aligned} f(x)&=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^{(4)}(1)}{4!}(x-1)^4+\dots \\ &=-1-\frac{2}{1!}(x-1)+\frac{6}{2!}(x-1)^2+\frac{24}{3!}(x-1)^3+\frac{24}{4!}(x-1)^4+\dots \\ &=-1-2(x-1)+3(x-1)^2+4(x-1)^3+1(x-1)^4+\dots \\ &=-1-2(x-1)+3(x-1)^2+4(x-1)^3+(x-1)^4+\dots \end{aligned}$$

Therefore, the required Taylor series is,

$$f(x)=\boxed{-1-2(x-1)+3(x-1)^2+4(x-1)^2+(x-1)^4}.$$

Since $f(x)=-1-2(x-1)+3(x-1)^2+4(x-1)^2+(x-1)^4$ is 4th degree polynomial in x, so it converges for all real values of x.

Hence, radius of convergence is $\boxed{R=\infty}$.

Q14E

Consider the following function:

$$f(x)=x-x^3$$

The objective is to find the Taylor series for $f(x)=x-x^3$ at centre $a=-2$ and also find the radius of convergence.

The Taylor series states that,

$$\begin{aligned} f(x)&=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n \\ &=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\dots \end{aligned}$$

Find the derivatives as follows:

$f(x)=x-x^3$	$f(-2)=6$
$f'(x)=1-3x^2$	$f'(-2)=-11$
$f''(x)=-6x$	$f''(-2)=12$
$f'''(x)=-6$	$f'''(-2)=-6$
$f^{(4)}(x)=0$	$f^{(4)}(-2)=0$
$f^{(5)}(x)=0$	$f^{(5)}(-2)=0$

Substitute the above tabular values in the Taylor series.

$$\begin{aligned}f(x)&=f(-2)+\frac{f'(-2)}{1!}(x+2)+\frac{f''(-2)}{2!}(x+2)^2+\frac{f'''(-2)}{3!}(x+2)^3+\dots\\&=6-\frac{11}{1!}(x+2)+12\frac{(x+2)^2}{2!}-6\frac{(x+2)^3}{3!}+0+0+\cdots\\&=6-11(x+2)+6(x+2)^2-(x+2)^3\end{aligned}$$

Hence the Taylor series of $f(x)=x-x^3$ with centre $a=-2$ is

$$\boxed{f(x)=6-11(x+2)+6(x+2)^2-(x+2)^3}.$$

Since $f(x)=x-x^3$ is a polynomial, and has finite terms, so it converges for all real values of x.

Hence, radius of convergence is $\boxed{R=\infty}$

Q15E

To find the **Taylor series** of a function centered at $a=2$, consider the function

$$f(x)=\ln x$$

Taylor series of the function f centered at a is defined as

$$\begin{aligned}f(x)&=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n\\&=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{iv}(a)}{4!}(x-a)^4+\dots\end{aligned}$$

So, it is need to compute $f(a),f'(a),f''(a),f'''(a),f^{iv}(a)\dots$

As $f(x) = \ln x$,

$f(2) = \ln 2$

So, $f(2) = \ln 2$

As $f(x) = \ln x$, on differentiating it with respect to x

$f'(x) = \frac{1}{x}$ Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$

So, $f'(2) = \frac{1}{2}$

As $f'(x) = \frac{1}{x}$, on differentiating it with respect to x

$f''(x) = -\frac{1}{x^2}$ Since $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

So, $f''(2) = -\frac{1}{4}$

As $f''(x) = -\frac{1}{x^2}$, on differentiating it with respect to x

$f'''(x) = \frac{2}{x^3}$ Since $\frac{d}{dx}\left(-\frac{1}{x^2}\right) = \frac{2}{x^3}$

So, $f'''(2) = \frac{2}{2^3}$

As $f'''(x) = \frac{2}{x^3}$, on differentiating it with respect to x

$f^{iv}(x) = \frac{-6}{x^4}$ Since $\frac{d}{dx}\left(\frac{2}{x^3}\right) = \frac{-6}{x^4}$

So, $f^{iv}(2) = \frac{-6}{2^4}$ and this pattern repeats indefinitely.

Therefore, Taylor series of the function $f(x)=\ln x$ centered at $a=2$ is

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{iv}(a)}{4!}(x-a)^4+\ldots$$

$$=\ln 2+\frac{\frac{1}{2}}{1!}(x-2)+\frac{\left(-\frac{1}{4}\right)}{2!}(x-2)^2+\frac{\left(\frac{2}{2^3}\right)}{3!}(x-2)^3+\frac{\left(\frac{-6}{2^4}\right)}{4!}(x-2)^4+\ldots.$$

The sum of the series

$$\ln 2+\frac{\frac{1}{2}}{1!}(x-2)+\frac{\left(-\frac{1}{4}\right)}{2!}(x-2)^2+\frac{\left(\frac{2}{2^3}\right)}{3!}(x-2)^3+\frac{\left(\frac{-6}{2^4}\right)}{4!}(x-2)^4+\ldots.$$

can be written as

$$=\ln 2+\frac{1}{2\cdot 1}(x-2)-\frac{1}{4\cdot 2}(x-2)^2+\frac{1}{3\cdot 2^3}(x-2)^3-\frac{1}{4\cdot 2^4}(x-2)^4+\ldots.$$

$$=\ln 2+\frac{(-1)^{1+1}}{1\cdot 2^1}(x-2)^1+\frac{(-1)^{2+1}}{2\cdot 2^2}(x-2)^2+\frac{(-1)^{3+1}}{3\cdot 2^3}(x-2)^3+\frac{(-1)^{4+1}}{4\cdot 2^4}(x-2)^4+\ldots.$$

$$=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot 2^n}(x-2)^n$$

Hence, Taylor series of the function $f(x)=\ln x$ centered at $a=2$ is

$$\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot 2^n}(x-2)^n$$

To find the **radius of convergence**,

let $a_n=\dfrac{(-1)^{n+1}}{n.2^n}$

Recall that, for the power series $\sum a_n(x-a)^n$, define radius of convergence R as

$$R=\frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

If $R=0$, the series is nowhere convergent and if $R=\infty$, the series is everywhere convergent and if $0<R<\infty$, the series converges absolutely for $|x-a|<R$, and diverges for $|x-a|>R$ that is, it has radius of convergence R

Compute limit of $\left|\frac{a_{n+1}}{a_n}\right|$ as $n \to \infty$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n.2^n}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n}{2(n+1)}$$

$$= \lim_{n\to\infty}\frac{1}{2\left(1+\frac{1}{n}\right)} \to \frac{1}{2} \quad \text{as } n \to \infty$$

Q16E

Taylor's series centered at 'a ' is $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

Given function $f(x) = \frac{1}{x}, a = -3$

$f'(x) = \frac{-1}{x^2}, f''(x) = \frac{2}{x^3}, f'''(x) = \frac{-3!}{x^4}, f^{iv}(x) = \frac{4!}{x^5},$

In general, $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$ ...... (1)

From (1), we get $f^{(n)}(-3) = \frac{(-1)^n n!}{(-3)^{n+1}} = \frac{-n!}{3^{n+1}}$

The Tayler series for $f(x)$ at a = -3 is

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{n=0}^{\infty}\frac{-n!}{(3)^{n+1}n!}(x+3)^n$$

$$= \sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}(x+3)^n$$

To find the radius of converges we let $a_n = \frac{(-1)}{3^{n+1}}(x+3)^n$

Then keeping the ratio test in view,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{3^{n+2}}(x+3)^{n+1}}{\frac{1}{3^{n+1}}(x+3)^n}\right|$$

$$= \frac{1}{3}|x+3|$$

The series converges if $\frac{|x+3|}{3} < 1$,

That is $|x+3| < 3$.

Thus, the radius of converges is $R = 3$.

Q17E

To find the **Taylor series** of a function centered at $a=3$, consider the function

$f(x)=e^{2x}$

Taylor series of the function f centered at a is defined as

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$
$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^{2}+\frac{f'''(a)}{3!}(x-a)^{3}+\frac{f^{iv}(a)}{4!}(x-a)^{4}+\ldots$$

So, it is need to compute $f(a), f'(a), f''(a), f'''(a), f^{iv}(a)\ldots$

As $f(x)=e^{2x}$,

$$f(3)=e^{2(3)}$$
$$=e^{6}$$

So, $f(3)=e^{6}$

As $f(x)=e^{2x}$, on differentiating it with respect to x

$f'(x)=2e^{2x}$ Since $\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$

So, $f'(3)=2e^{6}$

As $f'(x)=2e^{2x}$, on differentiating it with respect to x

$f''(x)=4e^{2x}$

So, $f''(3)=4e^{6}$

As $f''(x)=4e^{2x}$, on differentiating it with respect to x

$f'''(x)=8e^{2x}$

So, $f'''(3)=8e^{6}$

As $f'''(x)=8e^{2x}$, on differentiating it with respect to x

$f^{iv}(x)=16e^{2x}$

So, $f^{iv}(3)=16e^{6}$ and this pattern repeats indefinitely.

Therefore, Taylor series of the function $f(x)=e^{2x}$ centered at $a=3$ is

$$
\begin{aligned}
f(x) &= f(3)+\frac{f'(3)}{1!}(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3+\frac{f^{iv}(3)}{4!}(x-3)^4+\dots \\
&= e^6+\frac{2e^6}{1!}(x-3)+\frac{4e^6}{2!}(x-3)^2+\frac{8e^6}{3!}(x-3)^3+\frac{16e^6}{4!}(x-3)^4+\dots. \\
&= \frac{2^0e^6}{0!}(x-3)^0+\frac{2^1e^6}{1!}(x-3)^1+\frac{2^2e^6}{2!}(x-3)^2+\frac{2^3e^6}{3!}(x-3)^3+\frac{2^4e^6}{4!}(x-3)^4+\dots. \\
&= \sum_{n=0}^{\infty}\frac{2^n e^6}{n!}(x-3)^n
\end{aligned}
$$

To find the radius of convergence of $f(x)=e^{2x}$,

consider its power series representation

$$\sum_{n=0}^{\infty}\frac{2^n e^6}{n!}(x-3)^n$$

Recall that, for the power series $\sum a_n(x-a)^n$, define radius of convergence R as

$$R=\frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

If $R=0$, the series is nowhere convergent and if $R=\infty$, the series is everywhere convergent and if $0<R<\infty$, the series converges absolutely for $|x-a|<R$, and diverges for $|x-a|>R$ that is, it has radius of convergence R

For the series $\sum_{n=0}^{\infty}\frac{2^n e^6}{n!}(x-3)^n$, let $a_n=\frac{2^n e^6}{n!}$

Compute limit of $\left|\frac{a_{n+1}}{a_n}\right|$ as $n\to\infty$

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}e^6}{(n+1)!}\times\frac{n!}{2^n e^6}\right| \\
&= \lim_{n\to\infty}\frac{2}{n+1} \\
&= \frac{2}{\infty} \qquad \text{As } n\to\infty,\ (n+1)\to\infty \\
&= 0
\end{aligned}
$$

As $R=\dfrac{1}{\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|}$ and $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=0$, it follows that

$$R=\frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$
$$=\frac{1}{0}$$
$$=\infty$$

Hence, the series has radius of convergence $\boxed{\infty}$.

v

Q18E

We have
$$\begin{aligned} f(x)&=\cos x & f(\pi)&=\cos\pi=-1\\ f'(x)&=-\sin x & f'(\pi)&=-\sin\pi=0\\ f''(x)&=-\cos x & f''(\pi)&=-\cos\pi=-(-1)=1\\ f'''(x)&=\sin x & f'''(x)&=\sin\pi=0\\ f^{(iv)}(x)&=\cos x & f^{(iv)}(\pi)&=\cos\pi=-1 \end{aligned}$$
....

....

The Taylor series at $a=\pi$ is

$$f(\pi)+\frac{f'(\pi)(x-\pi)}{1!}+\frac{f''(\pi)(x-\pi)^2}{2!}+\frac{f'''(\pi)(x-\pi)^3}{3!}+\frac{f^{(iv)}(\pi)(x-\pi)^4}{4!}+\ldots\ldots$$
$$=(-1)+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}+\ldots\ldots.$$
$$=\boxed{\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}}$$

Let $\quad a_n=\dfrac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}$

Then $\quad \lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+2}(x-\pi)^{2n+2}}{(2n+2)!}\times\dfrac{(2n)!}{(-1)^{n+1}(x-\pi)^{2n}}\right|$

$$=\lim_{n\to\infty}\frac{|x-\pi|^2}{(2n+2)(2n+1)}=0<1\quad\text{for all x}$$

Thus the radius of convergence is $\boxed{R=\infty}$

Q19E

We have
$$\begin{aligned} f(x)&=\sin x & f(\pi/2)&=\sin(\pi/2)=1\\ f'&=\cos x & f'(\pi/2)&=\cos(\pi/2)=0\\ f''&=-\sin x & f''(\pi/2)&=-\sin(\pi/2)=-1\\ f'''&=-\cos x & f'''(\pi/2)&=-\cos(\pi/2)=0\\ f^{(iv)}&=\sin x & f^{(iv)}(\pi/2)&=\sin(\pi/2)=1 \end{aligned}$$
....

....

The Taylor series at $a=\pi/2$ is

$$f\left(\frac{\pi}{2}\right)+\frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right)+\frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3+\frac{f^{(iv)}\left(\frac{\pi}{2}\right)}{4!}\left(x-\frac{\pi}{2}\right)^4+\ldots\ldots$$

$$=1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4+\ldots\ldots$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}}$$

Let $\quad a_n=\dfrac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}$

Then $\quad \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\dfrac{(x-\pi/2)^{2n+2}}{(2n+2)!}\times\dfrac{(2n)!}{(x-\pi/2)^{2n}}\right|$

$$=\lim_{n\to\infty}\left|\frac{(x-\pi/2)^2}{(2n+2)(2n+1)}\right|=0<1 \quad \text{for all x}$$

Thus by the ratio test, the radius of convergence is $\boxed{R=\infty}$

Q20E

To find the **Taylor series** of a function centered at $a=16$, consider the function

$f(x)=\sqrt{x}$

Taylor series of the function f centered at a is defined as

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$$
$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{iv}(a)}{4!}(x-a)^4+\ldots$$

So, it is need to compute $f(a),f'(a),f''(a),f'''(a),f^{iv}(a)\ldots$

As $f(x)=\sqrt{x}$,

$$f(16)=\sqrt{16}$$
$$=4$$

So, $f(16)=4$

As $f(x)=\sqrt{x}$, on differentiating it with respect to x

$$f'(x)=\frac{d}{dx}\left(\sqrt{x}\right)$$
$$=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$
$$=\frac{1}{2}x^{\frac{1}{2}-1} \qquad \text{using } \frac{d}{dx}\left(x^n\right)=nx^{n-1}$$
$$=\frac{1}{2}x^{-\frac{1}{2}}$$
$$=\frac{1}{2\sqrt{x}}$$

So, $f'(16)=\dfrac{1}{2\sqrt{16}}$

$$=\frac{1}{2\cdot 4}$$

As $f'(x)=\dfrac{1}{2\sqrt{x}}$, on differentiating it with respect to x

$$f''(x)=\frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \qquad \text{since } \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$$
$$=\frac{1}{2}\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$
$$=\frac{1}{2}\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$
$$=-\frac{1}{4}x^{-\frac{3}{2}}$$
$$=-\frac{1}{(2)^2 x^{\frac{3}{2}}}$$

So, $f''(16) = -\dfrac{1}{(2)^2 (16)^{\frac{3}{2}}}$

$$= -\frac{1}{(2)^2 (4^2)^{\frac{3}{2}}}$$

$$= -\frac{1}{(2)^2 \cdot 4^3}$$

As $f''(x) = -\dfrac{1}{(2)^2 x^{\frac{3}{2}}}$, on differentiating it with respect to x

$$f'''(x) = \frac{d}{dx}\left(-\frac{1}{(2)^2 x^{\frac{3}{2}}}\right)$$

$$= -\frac{1}{(2)^2}\frac{d}{dx}\left(x^{-\frac{3}{2}}\right)$$

$$= -\frac{1}{(2)^2}\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right)$$

$$= \frac{3}{(2)^3}x^{-\frac{5}{2}}$$

$$= \frac{3}{(2)^3 x^{\frac{5}{2}}}$$

So, $f'''(16) = \dfrac{3}{(2)^3 (16)^{\frac{5}{2}}}$

$$= \frac{3}{(2)^3 (4^2)^{\frac{5}{2}}}$$

$$= \frac{3}{(2)^3 \cdot 4^5}$$

As $f'''(x)=\frac{3}{(2)^3 x^{\frac{5}{2}}}$, on differentiating it with respect to x

$$f^{iv}(x)=\frac{d}{dx}\left(\frac{3}{(2)^3 x^{\frac{5}{2}}}\right)$$

$$=\frac{3}{(2)^3}\frac{d}{dx}\left(x^{-\frac{5}{2}}\right)$$

$$=\frac{3}{(2)^3}\left(-\frac{5}{2}x^{-\frac{5}{2}-1}\right)$$

$$=-\frac{3\cdot 5}{(2)^4}x^{-\frac{7}{2}}$$

$$=-\frac{3\cdot 5}{(2)^4 x^{\frac{7}{2}}}$$

So, $f^{iv}(16)=-\frac{3\cdot 5}{(2)^4(16)^{\frac{7}{2}}}$

$$=-\frac{3\cdot 5}{(2)^4\left(4^2\right)^{\frac{7}{2}}}$$

$$=-\frac{3\cdot 5}{(2)^4(4)^7}$$

and this pattern repeats indefinitely.

Therefore, Taylor series of the function $f(x)=\sqrt{x}$ centered at $a=16$ is

$$f(x)=f(16)+\frac{f'(16)}{1!}(x-16)+\frac{f''(16)}{2!}(x-16)^2+\frac{f'''(16)}{3!}(x-16)^3$$
$$+\frac{f^{iv}(16)}{4!}(x-16)^4+\ldots$$

$$=4+\frac{\frac{1}{2\cdot 4}}{1!}(x-16)+\frac{-\frac{1}{(2)^2\cdot 4^3}}{2!}(x-16)^2+\frac{\frac{3}{(2)^3\cdot 4^5}}{3!}(x-16)^3+$$
$$\frac{-\frac{3\cdot 5}{(2)^4(4)^7}}{4!}(x-16)^4+\ldots.$$

$$=4+\frac{(x-16)}{8}+\frac{(-1)^{2-1}}{(2)^2\cdot\left(2^2\right)^3\cdot 2!}(x-16)^2+\frac{(-1)^{3-1}3}{(2)^3\cdot\left(2^2\right)^5\cdot 3!}(x-16)^3+$$
$$\frac{(-1)^{4-1}3\cdot 5}{(2)^4\cdot\left(2^2\right)^7\cdot 4!}(x-16)^4+\ldots.$$

$$=4+\frac{(x-16)}{8}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}3\cdot 5\cdots(2n-3)}{2^{5n-2}n!}(x-16)^n$$

To find the **radius of convergence** of $f(x)=\sqrt{x}$,

consider its power series representation

$$\sum_{n=2}^{\infty}\frac{(-1)^{n-1}3\cdot 5\cdots(2n-3)}{2^{5n-2}n!}(x-16)^n$$

Recall that, for the power series $\sum a_n(x-a)^n$, define radius of convergence R as

$$R=\frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

If $R=0$, the series is nowhere convergent and if $R=\infty$, the series is everywhere convergent and if $0<R<\infty$, the series converges absolutely for $|x-a|<R$, and diverges for $|x-a|>R$ that is, it has radius of convergence R

For the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} 3\cdot5\cdots(2n-3)}{2^{5n-2} n!}(x-16)^n$,

let $a_n = \frac{(-1)^{n-1} 3\cdot5\cdots(2n-3)}{2^{5n-2} n!}$

Compute limit of $\left|\frac{a_{n+1}}{a_n}\right|$ as $n \to \infty$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1-1} 3\cdot5\cdots(2(n+1)-3)}{2^{5(n+1)-2}(n+1)!} \times \frac{2^{5n-2} n!}{(-1)^{n-1} 3\cdot5\cdots(2n-3)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3\cdot5\cdots(2n-3)(2n-1)}{2^{5n+3} n+1} \cdot \frac{2^{5n-2}}{3\cdot5\cdots(2n-3)}\right|$$

$$= \lim_{n\to\infty}\frac{2^{-2}(2n-1)}{2^3(n+1)}$$

$$= \lim_{n\to\infty}\frac{2^{-2}\left(2-\frac{1}{n}\right)}{2^3\left(1+\frac{1}{n}\right)}$$

$$= \frac{2^{-2}(2-0)}{2^3(1+0)} \qquad \text{As } n\to\infty, \frac{1}{n}\to 0$$

$$= \frac{1}{16}$$

As $R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{16}$, it follows that

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

$$= \frac{1}{\left(\frac{1}{16}\right)}$$

$$= 16$$

Hence, the series has radius of convergence $\boxed{16}$

Q21E

First find the Maclaurin series for $f(x) = \sin \pi x$ as shown below:

$$f(x) = \sin(\pi x) \quad f(0) = \sin(0) = 0$$

$$f'(x) = \pi \cos(\pi x) \quad f'(0) = \pi \cos(0) = \pi$$

$$f''(x) = -\pi^2 \sin(\pi x) \quad f''(0) = -\pi^2 \sin(0) = 0$$

$$f'''(x) = -\pi^3 \cos(\pi x) \quad f'''(0) = -\pi^3 \cos(0) = -\pi^3$$

$$f^{(4)}(x) = \pi^4 \sin(\pi x) \quad f^{(4)}(0) = \pi^4 \sin(0) = 0$$

$$f^{(5)}(x) = \pi^5 \cos(\pi x) \quad f^{(5)}(0) = \pi^5 \cos(0) = \pi^5$$

The Maclaurin series for $f(x) = \sin \pi x$ is,

$$\begin{aligned} f(x) &= \sin(\pi x) \\ &\approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \end{aligned}$$

$$= 0 + \pi x + \frac{0}{2!}x^2 + \frac{(-\pi^3)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{\pi^5}{5!}x^5 + \dots$$

$$= \pi x + \frac{(-\pi^3)}{3!}x^3 + \frac{\pi^5}{5!}x^5 + \dots$$

$$= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 + \dots$$

It can be written as $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!}$.

Since $-1 \le \sin \pi x \le 1$ and $-1 \le \cos \pi x \le 1$, so $\left|f^{(n+1)}(x)\right| \le \pi^{n+1}$ for all x.

So take $M = \pi^{n+1}$ in Taylor's inequality:

$$\begin{aligned} |R_n(x)| &\le \frac{M}{(n+1)!}\left|x^{n+1}\right| \\ &= \frac{\pi^{n+1}}{(n+1)!}\left|x^{n+1}\right| \\ &= \frac{\pi^{n+1}}{(n+1)!}\left|(x)^{n+1}\right| \end{aligned}$$

Take limit on both sides.

$$\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{\pi^{n+1}}{(n+1)!}|(x)^{n+1}|$$

$$\le \lim_{n\to\infty}\left[\frac{1}{(n+1)!}\cdot\left(\pi^{n+1}|(x)^{n+1}|\right)\right]$$

$$\le 0\cdot\left(\pi^{n+1}|(x)^{n+1}|\right) \qquad \left[\text{Since as } n\to\infty, \frac{1}{(n+1)!}\to 0\right]$$

$$\le 0$$

Also, $0 \le |R_n(x)|$, by squeeze theorem, $\lim_{n\to\infty}|R_n(x)| = 0$.

That implies $R_n(x)\to 0$ as $n\to\infty$.

Therefore, $\sin(\pi x)$ is equal to the sum of its Maclaurin series for all x.

Q22E

We have $f(x) = \sin x \qquad f(\pi/2) = \sin(\pi/2) = 1$

$$f' = \cos x \qquad f'(\pi/2) = \cos(\pi/2) = 0$$

$$f'' = -\sin x \qquad f''(\pi/2) = -\sin(\pi/2) = -1$$

$$f''' = -\cos x \qquad f'''(\pi/2) = -\cos(\pi/2) = 0$$

$$f^{(iv)} = \sin x \qquad f^{(iv)}(\pi/2) = \sin(\pi/2) = 1$$

....

....

The Taylor series at $a = \pi/2$ is

$$f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3 + \frac{f^{(iv)}\left(\frac{\pi}{2}\right)}{4!}\left(x-\frac{\pi}{2}\right)^4 + \ldots\ldots$$

$$= 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 + \ldots\ldots$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}}$$

Since $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$

Then $f^{(n+1)}\left(\frac{\pi}{2}\right)$ is ± 1 *or* 0

So $\left|f^{(n+1)}\left(\frac{\pi}{2}\right)\right| \le 1$ for all x so we take $M = 1$

In Taylor's inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|(x-\pi/2)^{n+1}|$$

$$= \frac{|(x-\pi/2)^{n+1}|}{(n+1)!} \to 0 \quad \text{as} \quad n\to\infty$$

Then $R_n(x)\to 0$ as $n\to\infty$. So $\sin x$ is equal to its Taylor series for all x.

In Taylor's inequality

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|(x-\pi/2)^{n+1}\right|$$

$$= \frac{\left|(x-\pi/2)^{n+1}\right|}{(n+1)!} \to 0 \quad \text{as} \qquad n \to \infty$$

Then $R_n(x) \to 0$ as $n \to \infty$. So $\sin x$ is equal to its Taylor series for all x.

Q23E

The Maclaurin's series for sinh x is

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Since $f^{(n+1)}(x) = \sinh x \qquad or \qquad \cosh x$

So $\left|f^{(n+1)}(x)\right| \le 1$ for all x .

We can take $M = 1$ in Taylor's inequality

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}|x|^{n+1}$$

$$= \frac{1}{(n+1)!}\left|x^{n+1}\right| \to 0 \qquad as \qquad n \to \infty$$

Then $R_n(x) \to 0 \qquad as \qquad n \to \infty$

Hence sinh x is equal to its Maclaurin's series for all x.

Q24E

Consider the Maclaurin series for $\cosh x$ is $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Here we need to prove this series represents $\cosh x$ for all x.

Let $f(x)=\frac{x^{2n}}{(2n)!}$.

$$f'(x)=\frac{2nx^{2n-1}}{(2n)!}$$
$$=\frac{x^{2n-1}}{(2n-1)!}$$

And

$$f''(x)=\frac{x^{2n-2}}{(2n-2)!}$$

Similarly,

$$f^{(n)}(x)=\frac{x^{2n-n}}{(2n-n)!}$$
$$=\frac{x^{n}}{(n)!}$$

And also

$$f^{(n+1)}(x)=\frac{x^{n+1}}{(n+1)!}.$$

The derivative of $\cosh x$ is $\sinh x$, and the second derivative of $\cosh x$ is $\cosh x$ and so on.

Therefore the $(n+1)^{\text{th}}$ derivative of $\cosh x$ is $\sinh x$ or $\cosh x$.when n is even the derivative is $\cosh x$ and when n is odd the derivative is $\sinh x$.

From the Taylor's inequality, $\left|f^{(n+1)}(x)\right|\le M$, here we can take $M=1$, which the positive number.

and from the Taylor's remainder,

$$\begin{aligned}|R_n(x)| &\le \frac{M}{(n+1)!}|x^{n+1}| \\ &\le \frac{1}{(n+1)!}|x^{n+1}| \\ &\le \frac{|x^{n+1}|}{(n+1)!}\end{aligned}$$

As $n \to \infty$, the limit of the above inequality becomes as

$$\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\left[\frac{|x^{n+1}|}{(n+1)!}\right].$$

Now we use the ratio test to find the value of the limit of the Taylor's remainder,

$$\lim_{n\to\infty}|R_n(x)|.$$

Let $a_n = \dfrac{|x^{n+1}|}{(n+1)!}$,

and

$$\begin{aligned}a_{n+1} &= \frac{|x^{n+1+1}|}{(n+1+1)!} \\ &= \frac{|x^{n+2}|}{(n+2)!}\end{aligned}$$

Now by ratio test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+2}}{(n+2)!}\times\frac{(n+1)!}{x^{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+2}\right| \\ &= x\cdot\lim_{n\to\infty}\left|\frac{1}{n+2}\right| \\ &= 0\end{aligned}$$

Thus the series converges to 0.

Therefore, $R_n(x) \to 0$ as $n \to \infty$.

Hence, the Maclaurin series $\sum_{n=0}^{\infty}\dfrac{x^{2n}}{(2n)!}$ represents $\cosh x$ for all x.

Q25E

Consider the function,

$f(x)=\sqrt[4]{1-x}$.

Rewrite the function as,

$f(x)=(1-x)^{\frac{1}{4}}$

Recall that, the binomial series expansion of $(1+x)^k$ is denoted as,

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty}\binom{k}{n}x^n \\ &= 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(2k-1)}{3!}x^3+\cdots\end{aligned}$$

Use binomial series expansion with $k=\frac{1}{4}$ and with x replaced by $-x$, then the expansion can be written as,

$$\begin{aligned}(1-x)^{\frac{1}{4}} &= 1-\frac{1}{4}x+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}x^2-\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}x^3+\cdots \\ &= 1-\frac{x}{4}+\frac{\left(\frac{-3}{4^2}\right)}{2!}x^2+\frac{\left(\frac{21}{64}\right)}{3!}x^3+\cdots \\ &= 1-\frac{x}{4}-\sum_{n=2}^{\infty}\frac{3\cdot 7\cdots(4n-1)}{4^n(n)!}x^n \\ &= 1-\frac{x}{4}-\sum_{n=2}^{\infty}\frac{3\cdot 7\cdots(4n-1)}{4^n(n)!}(x)^n\end{aligned}$$

Since the above binomial expansion is valid when $|x|<1$.

Hence, the radius of convergence is $R=1$.

Q26E

Use binomial series to expand the function $f(x)=\sqrt[3]{8+x}$ as a power series.

Write the binomial series for a function,

If k is any real number and $|x|<1$, then $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kn+\frac{k(k-1)}{2!}x^2+\ldots$

The binomial series converges when $|x|<1$. ......(1)

Consider $\sqrt[3]{8+x} = 8^{\frac{1}{3}}\left(1+\frac{x}{8}\right)^{\frac{1}{3}}$

$$= 2\left(1+\frac{1}{3}\left(\frac{x}{8}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{x}{8}\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{x}{8}\right)^3+\dots\right)$$

$$= 2\left(1+\frac{x}{3^1\cdot 8}+\frac{-2}{3^2\cdot 2!\cdot 64}x^2+\frac{(-2)(-5)}{3^3\cdot 3!\cdot 512}x^3+\dots\right)$$

$$= 2\left(\begin{array}{l} 1+\frac{x}{3^1\cdot 1!\cdot 8}-\frac{2}{3^2\cdot 2!\cdot 64}x^2+\frac{2\cdot 5}{3^3\cdot 3!\cdot 512}x^3+\dots \\ +(-1)^{n-1}\frac{2.5.8\dots(3n-4)}{n!3^n8^n}x^n\ (\text{for } n\ge 2)+\dots \end{array}\right)$$

From the fact (1), the series is converges for $|x|<1$,

So the radius of convergence is $\boxed{R = 1}$.

Q27E

Consider the function $f(x)=\frac{1}{(2+x)^3}$.

The objective is to expand the above function as a power series using the binomial series.

And also state the radius of convergence.

Recollect the **Binomial series**:

If *k* is any real number and $|x|<1$, then,

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots \quad \text{...... (1)}$$

Rewrite the function $f(x)$ as,

$$f(x)=\frac{1}{(2+x)^3} = \frac{1}{2^3\left(1+\frac{x}{2}\right)^3} = \frac{1}{2^3}\left(1+\frac{x}{2}\right)^{-3}$$

Use the binomial series (1) with $k=-3$ and with x replaced by $\frac{x}{2}$ to get,

$$\begin{aligned}
f(x) &= \frac{1}{(2+x)^3} \\
&= \frac{1}{2^3}\left(1+\frac{x}{2}\right)^{-3} \\
&= \frac{1}{2^3}\left[\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n\right] \qquad \text{Use Binomial series.} \quad \cdots\cdots(2)
\end{aligned}$$

$$=\frac{1}{2^3}\left[1+(-3)\left(\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-3)(-3-1)(-3-2)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$=\frac{1}{2^3}\left[1-\frac{3}{2}\cdot x+\frac{(-3)(-4)}{2^2\cdot 2!}\cdot x^2+\frac{(-3)(-4)(-5)}{2^3\cdot 3!}\cdot x^3+\cdots\right]$$

$$=\frac{1}{2^3}\left[1-\frac{3}{2}\cdot x+\frac{3\cdot 4}{2^2\cdot 2!}\cdot x^2-\frac{3\cdot 4\cdot 5}{2^3\cdot 3!}\cdot x^3+\cdots\right]$$

$$=\frac{1}{2^3}\left(1-\frac{3}{2}x+\frac{3\cdot 4}{2^2\cdot 2!}x^2-\frac{4\cdot 5}{2^3\cdot 2!}x^3+\ldots\right) \qquad \text{By cancellation}$$

$$=\frac{1}{2^3}\left(1-\frac{2\cdot 3}{2^2}x+\frac{3\cdot 4}{2^3}x^2-\frac{4\cdot 5}{2^4}x^3+\ldots\right)$$

$$=\frac{1}{2^3}\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2^{n+1}}x^n$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2^{n+4}}x^n \qquad \text{Rewrite the terms from } n=0$$

From the binomial series expansion (2), observe that this series converges when $\left|\frac{x}{2}\right|<1,$ that is, $|x|<2,$ so the radius of convergence is $R=2.$

Thus, the power series for the function $f(x)=\frac{1}{(2+x)^3}$ is $\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2^{n+4}}x^n}$ and the radius of convergence is $R=\boxed{2}.$

Q28E

Consider the function,

$f(x)=(1-x)^{2/3}$.

The objective is to find the power series expansion for this function and determine the radius of convergence.

Binomial series: If *k* is any real number and $|x|<1$, then

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots$$

The given function can be rewritten as,

$$\begin{aligned}f(x)&=(1+(-x))^{2/3}\\&=\sum_{n=0}^{\infty}\binom{2/3}{n}(-x)^n\\&=\sum_{n=0}^{\infty}\binom{2/3}{n}(-1)^n x^n\\&=1-\left(\frac{2}{3}\right)x+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}x^2-\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}x^3+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{4!}x^4+\ldots\end{aligned}$$

$$=1-\frac{2}{3}x-2\frac{1}{3^2\cdot 2!}x^2-2\frac{4}{3^3\cdot 3!}x^3-2\frac{4\cdot 7}{3^4\cdot 4!}x^4-2\frac{4\cdot 7\cdot 10}{3^5\cdot 5!}x^5+\ldots$$

$$=1-\frac{2}{3}x-2\sum_{n=2}^{\infty}\frac{4\cdot 7\cdots(3n-5)}{3^n\cdot n!}x^n$$

As the binomial series expansion $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n$ holds for $|x|<1$, the binomial series expansion of $f(x)=(1+(-x))^{2/3}=\sum_{n=0}^{\infty}\binom{2/3}{n}(-x)^n$ holds for

$$|-x|<1$$
$$|x|<1$$

Thus, the radius of convergence is $\boxed{R=1}$.

Therefore, the required power series representation for the given function

$f(x)=(1-x)^{2/3}$ is,

$\boxed{f(x)=(1-x)^{2/3}=1-\frac{2}{3}x-2\sum_{n=2}^{\infty}\frac{4\cdot 7\cdots(3n-5)}{3^n\cdot n!}x^n}$ with radius of convergence is $\boxed{R=1}$.

Q29E

Consider the function,

$f(x) = \sin \pi x$.

The objective is to determine the Maclaurin series of the function.

Use the Maclaurin series of $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, to obtain the Maclaurin series of the function.

So, replace x with πx in the Maclaurin series of $\sin x$, to get:

$$\sin \pi x = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} x^{2n+1}}$$

As the radius of convergence for $\sin x$ is $R = \infty$, for all x.

Therefore, the radius of convergence for the series $\sin \pi x$ is $\boxed{R = \infty}$.

Q30E

Consider the function,

$f(x) = \cos\left(\frac{\pi x}{2}\right)$

According to Table 1, the Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \quad \ldots\ldots (1)$$

Replace x with $\frac{\pi x}{2}$ in the series for $\cos x$, then

$$\cos\left(\frac{\pi x}{2}\right)=1-\frac{\left(\frac{\pi x}{2}\right)^2}{2!}+\frac{\left(\frac{\pi x}{2}\right)^4}{4!}-\frac{\left(\frac{\pi x}{2}\right)^6}{6!}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi x}{2}\right)^{2n}}{(2n)!}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{(\pi/2)^{2n}}{(2n)!}x^{2n}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{2^{2n}(2n)!}x^{2n}$$

Hence the required Maclaurin series expansion of the given function is:

$$\cos\left(\frac{\pi x}{2}\right)=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{2^{2n}(2n)!}x^{2n}}.$$

Q31E

Consider the function,

$$f(x)=e^x+e^{2x}$$

To expand the function as a Maclaurin series, use the following Maclaurin series.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \quad \text{...... (1)}$$

Replace x with $2x$ in the series for e^x, get

$$e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$$

$$=1+\frac{(2x)}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots \quad \text{...... (2)}$$

$$=1+2x+2x^2+\frac{4x^3}{3}+\cdots$$

Add equation (1) and (2), get

$$
\begin{aligned}
f(x) &= e^x + e^{2x} \\
&= \left[1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right]+\left[1+2x+2x^2+\frac{4x^3}{3}+\cdots\right] \\
&= \sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}+\frac{2^n x^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\frac{x^n}{n!}\left(1+2^n\right)
\end{aligned}
$$

Hence the required Maclaurin series expansion of the given function is:

$$e^x + e^{2x} = \boxed{\sum_{n=0}^{\infty}\frac{x^n}{n!}\left(1+2^n\right)}.$$

Q32E

Consider the function $f(x) = e^x + 2e^{-x}$.

To expand the function as a Maclaurin series, use the following Maclaurin series.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \quad \text{......} \ (1)$$

Replace x with $-x$ in the series for e^x, get

$$
\begin{aligned}
e^{-x} &= \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} \\
&= 1+\frac{(-x)}{1!}+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\cdots \quad \text{......} \ (2) \\
&= 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots
\end{aligned}
$$

Multiply with 2 on both sides of equation (2), get

$$
\begin{aligned}
2e^{-x} &= 2\left[1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right] \quad \text{......} \ (3) \\
&= 2-2x+\frac{2x^2}{2!}-\frac{2x^3}{3!}+\cdots
\end{aligned}
$$

Add equation (1) and (2), get

$$f(x) = e^x + 2e^{-x}$$
$$= \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + \left[2 - 2x + \frac{2x^2}{2!} - \frac{2x^3}{3!} + \cdots\right]$$
$$= 1 + 2 + \left(\frac{x}{1!} - 2x\right) + \left(\frac{x^2}{2!} + \frac{2x^2}{2!}\right) + \left(\frac{x^3}{3!} - \frac{2x^3}{3!}\right) + \cdots$$
$$= 3 - x + \frac{3x^2}{2!} - \frac{x^3}{3!} + \cdots$$
$$= 3 - x + \frac{3x^2}{2} - \frac{x^3}{6} + \cdots$$

Hence the required Maclaurin series expansion of the given function is

$$e^x + 2e^{-x} = \boxed{= 3 - x + \frac{3x^2}{2} - \frac{x^3}{6} + \cdots}.$$

Q33E

We have

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$
$$\Rightarrow \ln\left(1+x^3\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n}}{n}$$
$$\Rightarrow x^2 \ln\left(1+x^3\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n+2}}{n}$$

Hence

$$x^2 \ln\left(1+x^3\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n+2}}{n}$$

Q34E

Maclaurin series: $f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \ldots$

Given function is $f(x) = x\cos\left(\frac{x^2}{2}\right)$ …… (1)

From the referred table, we have $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Replacing x in this series with $\frac{x^2}{2}$, we get $\cos\left(\frac{x^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{2}\right)^{2n}}{(2n)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n}(2n)!}$$

Multiplying both sides with x, we get $\boxed{x\cos\left(\frac{x^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{2n}(2n)!}}$

Q35E

Maclaurin series: $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$

Given function $f(x)=\frac{x}{\sqrt{4+x^2}}$

We know that the Binomial series for $(1+x)^k$ is

$$(1+x)^k=\sum_{n=0}^{k}\binom{k}{n}x^n$$

Using this, we have $\frac{x}{\sqrt{4+x^2}}=\frac{x}{2\left(1+\left(\frac{x}{2}\right)^2\right)^{\frac{1}{2}}}$

$$=\frac{x}{2}\left(1+\left(\frac{x}{2}\right)^2\right)^{-\frac{1}{2}}$$

$$=\frac{x}{2}\left\{1+\frac{\left(\frac{-1}{2}\right)}{1!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^4+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^6+\ldots\right\}$$

$$=\frac{x}{2}\left\{1-\frac{1}{2}\left(\frac{x}{2}\right)^2+\frac{1\cdot 3}{2^2\cdot 2!}\left(\frac{x}{2}\right)^4-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\left(\frac{x}{2}\right)^6+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}\left(\frac{x}{2}\right)^8-\ldots\right\}$$

$$=\left\{\frac{x}{2}-\frac{1}{2}\left(\frac{x}{2}\right)^3+\frac{1\cdot 3}{2^2\cdot 2!}\left(\frac{x}{2}\right)^5-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\left(\frac{x}{2}\right)^7+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}\left(\frac{x}{2}\right)^9-\ldots\right\}$$

$$=\frac{x}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n\cdot n!}\left(\frac{x}{2}\right)^{2n+1}$$

Or, $\frac{x}{\sqrt{4+x^2}}=\boxed{\frac{x}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\,x^{2n+1}}{2^{3n+1}\cdot n!}}$

Q36E

Maclaurin series: $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$

Given function $f(x)=\frac{x^2}{\sqrt{2+x}}$

We know that the Binomial series for $(1+x)^k$ is

$$(1+x)^k=\sum_{n=0}^{k}\binom{k}{n}x^n$$

Using this, we have $\frac{x^2}{\sqrt{2+x}}=\frac{x^2}{\sqrt{2}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}}$

$$= \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$$

$$= \frac{x^2}{\sqrt{2}}\left\{1+\frac{\left(\frac{-1}{2}\right)}{1!}\left(\frac{x}{2}\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3+\dots\right\}$$

$$= \frac{x^2}{\sqrt{2}}\left\{1-\frac{1}{2}\left(\frac{x}{2}\right)+\frac{1\cdot 3}{2^2\cdot 2!}\left(\frac{x}{2}\right)^2-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\left(\frac{x}{2}\right)^3+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}\left(\frac{x}{2}\right)^4-\dots\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{x^2-\frac{1}{2}\left(\frac{x}{2}\right)^3+\frac{1\cdot 3}{2^2\cdot 2!}\left(\frac{x}{2}\right)^4-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\left(\frac{x}{2}\right)^5+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}\left(\frac{x}{2}\right)^6-\dots\right\}$$

$$= \frac{x^2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{2^n\cdot n!}\left(\frac{x}{2}\right)^{n+1}$$

Or, $\frac{x^2}{\sqrt{2+x}} = \boxed{\frac{x^2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)\,x^{n+1}}{2^{2n+1}\cdot n!}}$

By ratio test, this series converges if $\left|\frac{x}{2^2}\right|<1$

So, the radius of convergence is 4.

Q37E

We have $f(x)=\sin^2 x=\frac{1}{2}(1-\cos 2x)$

The Maclaurin series for cos2x is

$$\cos 2x = 1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\dots$$

Then Maclaurin series for

$$f(x)=\frac{1}{2}(1-\cos 2x)=\frac{1}{2}\left[1-1+\frac{(2x)^2}{2!}-\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}\dots\dots\right]$$

$$=\frac{1}{2}\left[2^2\frac{x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}\dots\dots\dots\dots\right]$$

$$=\frac{2x^2}{2!}-\frac{2^3x^4}{4!}+\frac{2^5x^6}{6!}\dots\dots\dots\dots$$

$$=\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}2^{2n-1}\frac{x^{2n}}{(2n)!}},\quad\boxed{R=\infty}$$

Q38E

$$f(x)=\begin{cases}\dfrac{x-\sin x}{x^3} & \text{if } x\neq 0\\[2mm] \dfrac{1}{6} & \text{if } x=0\end{cases}$$

Since the Maclaurin series for sin x is

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\ldots\ldots\ldots$$

$$x-\sin x=\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}\ldots\ldots\ldots\ldots$$

$$\frac{x-\sin x}{x^3}=\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}\ldots\ldots\ldots\ldots$$

When $x\neq 0$, *Then*

$$\frac{x-\sin x}{x^3}=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+3)!}}$$

When $x=0$, Then

$$\frac{x-\sin x}{x^3}=\frac{1}{3!}-0+0+\ldots\ldots=\frac{1}{6}$$

Q39E

We have $f(x)=\cos\left(x^2\right)$

Since $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ is Maclaurin series for $\cos x$

Replace x *by* x^2

$$\cos\left(x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^2\right)^{2n}}{(2n)!}$$

$$\Rightarrow\boxed{\cos\left(x^2\right)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}}\quad\text{for all } x$$

We have $a_n=(-1)^n\dfrac{x^{4n}}{(2n)!}$

Then $a_{n+1}=\dfrac{(-1)^{n+1}x^{4n+4}}{(2n+2)!}$

So

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\left(x^{4n+4}\right)}{(2n+2)!}\frac{(2n)!}{(-1)^n x^{4n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-1)x^4}{(2n+2)(2n+1)}\right|$$

$$=\left|x^4\right|\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)}=0<1$$

So series is convergent for all x and then radius of convergence is $\boxed{R=\infty}$

Now we graph the function *f(x)* and few Taylor polynomials, and see that as n increases, $T_n\ (x)$ be a better approximation for $f\ (x)$

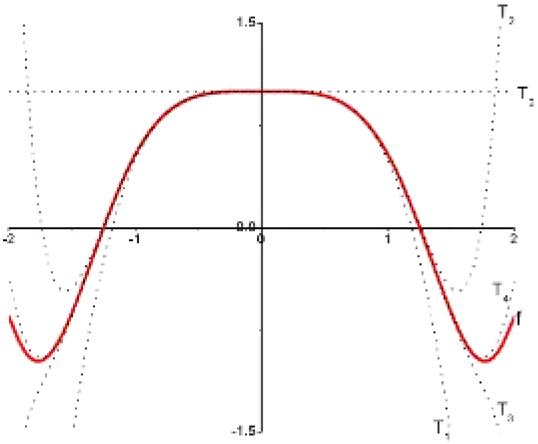


Fig.1

Q40E

We have $f\left(x\right)=e^{-x^2}+\cos x$

Maclaurin series for e^x is

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Replace x by $-x^2$

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{\left(-x^2\right)}{n!}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{n!}$$

And Maclaurin series for cos x is

$$\cos x=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{\left(2n\right)!}$$

So $$f\left(x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{n!}+\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{\left(2n\right)!}$$

$$\Rightarrow \boxed{f\left(x\right)=\sum_{n=0}^{\infty}\left(-1\right)^n x^{2n}\left(\frac{1}{n!}+\frac{1}{\left(2n\right)!}\right)} \qquad \textit{for all } x$$

Radius of convergence is $\boxed{R=\infty}$

Now we graph the function f (x) and T_n (x) for n =1, 2, 3, 4, ---and see that as n- increases, T_n (x) be a better approximation for f (x)

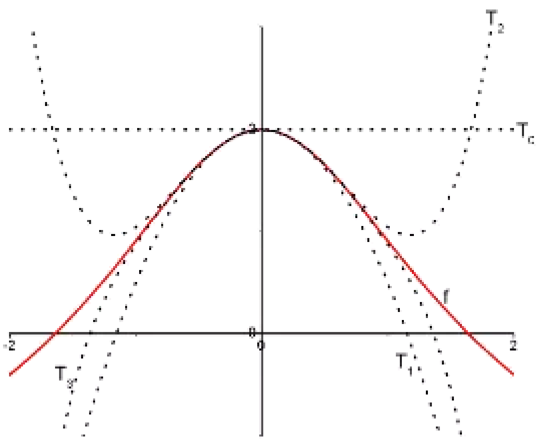


Fig.1

Q41E

Maclaurin series: $f(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \ldots + \dfrac{f^{(n)}(0)}{n!}x^n + \ldots$

Given function is xe^{-x}

Keeping the series in view, we find the derivatives of $f(x) = xe^{-x}$

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

$f''(x) = -e^{-x} - (1-x)e^{-x} = (-1)^1(2-x)e^{-x}$

Similarly, $f'''(x) = (-1)^2(3-x)e^{-x}, f^{iv}(x) = (-1)^3(4-x)e^{-x}, \ldots$

Putting $x = 0$ in the function and the derivatives, we get

$f(0) = 0, f'(0) = (1-0)e^{-0} = 1,$

$f''(0) = (-1)^1(2-0)e^{-0} = -2, f'''(0) = (-1)^2(3-0)e^{-0} = 3, \ldots$

Using these values in the expansion, we get

$$f(x) = xe^{-x} = 0 + \frac{1}{1!}x + \frac{-2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{-4}{4!}x^4 + \ldots$$

$$= x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \ldots$$

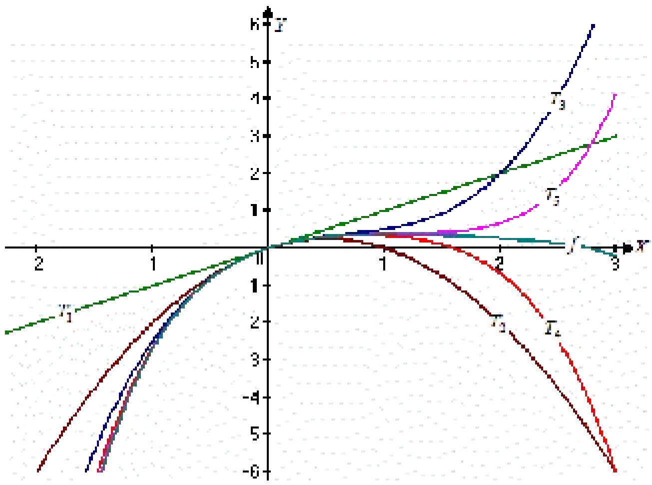
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(n-1)!}$$

The radius of convergence $R = \infty$

The first few terms of this series is $T_1(x) = x, T_2(x) = x - x^2, T_3(x) = x - x^2 + \dfrac{x^3}{2}$

$T_4(x) = x - x^2 + \dfrac{x^3}{2} - \dfrac{x^4}{6}$, $T_5(x) = x - x^2 + \dfrac{x^3}{2} - \dfrac{x^4}{6} + \dfrac{x^5}{24}$

The graph of these polynomials is shown in the following figure.



Q42E

Given that $f(x) = \tan^{-1}(x^3)$.

We know that the Maclaurin expansion of $\tan^{-1}(x)$ is given by $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

or $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ and the radius of convergence R is 1.

Replace x with x^3 in the expansion.

$$\tan^{-1}\left(x^3\right) = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \ldots$$

We get the radius of convergence as 1.

We have $T_1(x) = 0$, $T_2(x) = 0$, $T_3(x) = x^3$, $T_4(x) = x^3$, …, $T_8(x) = x^3$, $T_9\left(x\right) = x^3 - \dfrac{x^9}{3}$.

Therefore, the Maclaurin series of the function is $\boxed{x^5 - \dfrac{x^8}{2} + \dfrac{x^{11}}{3} - \dfrac{x^{13}}{4} + \ldots}$.

Q43E

We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$

$$\Rightarrow \cos\left(5^0\right) = 1 - 0.00381 + 0.0000024 + \ldots$$

$$\left(\begin{aligned} \because\ x &= 5^0 \\ &= 5 \times \frac{\pi}{180} \\ &= \frac{22}{252} \\ &= 0.0873 \end{aligned}\right)$$

$$= 0.99619$$

By the Alternating Series Estimation Theorem we know that

$$\left|s - s_3\right| \le b_4 \quad \text{where}\, b_4 = \frac{x^6}{6!}$$
$$= 0.00000000061.$$

So $\cos\left(5^0\right) \cong 0.99619$ is corrected up to five decimal places.

Q44E

We know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots.$$

$$\Rightarrow e^{-1/10} = 1 - \frac{1}{10} + \frac{1}{200} - \frac{1}{6000} + \frac{1}{24000} + \ldots.$$
$$= 1 - 0.1 + 0.005 - 0.00016 + 0.000041$$
$$= 0.904881$$

By the Alternating Series Estimation Theorem we know that

$$\left|s - s_5\right| \le b_6 \text{ where } b_6 = \frac{x^5}{5!}$$
$$= 0.000000834.$$

So $e^{-\frac{1}{10}} \cong 0.90488$ is corrected up to five decimal places.

Q45E

(a)

Consider the function,

$$f(x)=\frac{1}{\sqrt{1-x^2}}$$

Write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{1-x^2}}=\frac{1}{\left(1-x^2\right)^{1/2}}$$
$$=\left(1-x^2\right)^{-1/2}$$

Recollect the binomial series

$$(1+x)^p=\sum_{n=0}^{\infty}\binom{p}{n}x^n$$
$$=1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-3)}{3!}x^3+...,$$

where p is any real number and $|x|<1$

Using the binomial series with $p=-1/2$ and with x replaced by $-x^2$:

$$\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-1/2}$$
$$=\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-x^2\right)^n$$
$$=1+\left(-\frac{1}{2}\right)\left(-x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-x^2\right)^3$$
$$+...+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-x^2\right)^n+...$$
$$=1+\frac{1}{2^1 1!}x^2+\frac{1\cdot 3}{2^2 2!}x^4-\frac{1\cdot 3\cdot 5}{2^3 3!}x^6+...+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}x^{2n}+...$$
$$=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}x^{2n}$$

Therefore the binomial series for $f(x)$ is $\boxed{1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}x^{2n}}$.

(b)

Consider the function,

$$f(x)=\sin^{-1}(x)$$

Recollect the Maclaurin series for the function *f* states that

$$\begin{aligned} f(x)&=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\ &=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots \end{aligned}$$

Find the function *f* and all derivatives of the function *f* at 0 as follows:

$$f(x)=\sin^{-1}(x) \quad f(0)=0$$

$$f'(x)=\left(1-x^2\right)^{-1/2} \quad f'(0)=1$$

$$f''(x)=x\left(1-x^2\right)^{-3/2} \quad f''(0)=0$$

$$f'''(x)=3x^2\left(1-x^2\right)^{-5/2}+\left(1-x^2\right)^{-3/2} \quad f'''(0)=1$$

To find the Maclaurin series for the function $f(x)=\sin^{-1}(x)$:

$$\begin{aligned} f(x)&=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\ldots \\ &=0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{1}{3!}x^3+\frac{0}{4!}x^4+\frac{1}{5!}x^5+\frac{0}{6!}x^6+\frac{1}{7!}x^7\ldots \\ &=x+\frac{1}{3\cdot 2^1 1!}x^3+\frac{1\cdot 3\cdot 5}{5\cdot 2^2 2!}x^5+\ldots+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)2^n n!}x^{2n+1}+\ldots \\ &=x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)2^n n!}x^{2n+1} \end{aligned}$$

Therefore the Maclaurin series for the function $f(x)=\sin^{-1}(x)$ is

$$\boxed{x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)2^n n!}x^{2n+1}}.$$

To find the Maclaurin series for the function $f(x)=\sin^{-1}(x)$:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\dots$$
$$=0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{1}{3!}x^3+\frac{0}{4!}x^4+\frac{1}{5!}x^5+\frac{0}{6!}x^6+\frac{1}{7!}x^7\dots$$
$$=x+\frac{1}{3\cdot 2^1 1!}x^3+\frac{1\cdot 3\cdot 5}{5\cdot 2^2 2!}x^5+\dots+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)2^n n!}x^{2n+1}+\dots$$
$$=x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)2^n n!}x^{2n+1}$$

Therefore the Maclaurin series for the function $f(x)=\sin^{-1}(x)$ is

$$\boxed{x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)2^n n!}x^{2n+1}}.$$

v

Q46E

(a) $$\frac{1}{\sqrt[4]{1+x}}=\frac{1}{(1+x)^{\frac{1}{4}}}$$
$$=(1+x)^{-\frac{1}{4}}$$
$$=1-\frac{\frac{1}{4}x}{1!}+\frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)}{2!}x^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)}{3!}x^3+\dots\dots\dots\dots$$
$$=1-\frac{\left(\frac{1}{4}\right)x}{1!}+\frac{\left(\frac{1}{4}\right)\left(\frac{1.5}{4}\right)}{2!}x^2+\frac{\frac{1\cdot 5\cdot 9}{(4)^3}}{3!}x^3+\dots\dots\dots\dots$$
$$\Rightarrow\frac{1}{\sqrt[4]{1+x}}=1-\frac{1}{4\cdot 1!}(x)+\frac{1\cdot 5}{4^2\cdot 2!}(x^2)-\frac{1\cdot 5\cdot 9}{4^3\cdot 3!}(x^3)+\dots\dots\dots\dots \quad\rightarrow(1)$$

Which is the required power series expansion.

(b) $$\frac{1}{\sqrt[4]{1\cdot 1}}=\frac{1}{\sqrt[4]{1+0.1}}$$
$$=1-\frac{1}{4\cdot 1!}(0.1)+\frac{1.5}{4^2\cdot 2!}(0.1)^2-\frac{1\cdot 5\cdot 9}{4^3\cdot 3!}(0.1)^3+\dots\dots\dots\dots$$
$$=1-0.025+(0046875)(0.01)-(0.1171875)(0.001)+\dots\dots\dots$$
$$=1.00046875-0.025117187$$
$$=0.975351562$$
$$\square\ 0.975$$

Q47E

We have to evaluate $\int x\cos(x^3)dx$

The Maclaurin series for $x\cos(x^3)$ is

$$x\cos(x^3)=x\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!}$$

Or $$x\cos(x^3)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}$$

Now we integrate both sides

$$\int x\cos\left(x^3\right)dx=\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}dx$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int x^{6n+1}dx$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{x^{6n+2}}{(6n+2)}+C}$$

Q48E

We have to evaluate $\int\frac{e^x-1}{x}dx$

$$\frac{e^x-1}{x}=\frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\right)-1}{x}$$

$$=\frac{1}{x}\left[\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\right]$$

$$=\frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\ldots\ldots\ldots$$

Then

$$\int\frac{e^x-1}{x}dx=\frac{x}{1!}+\frac{x^2}{2.2!}+\frac{x^3}{3.3!}+\ldots\ldots\ldots+C=\boxed{\sum_{n=1}^{\infty}\frac{x^n}{(n)n!}+C}$$

Q49E

We know that the Maclaurin series for cos x is $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

$$\Rightarrow\frac{\cos x-1}{x}=\frac{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n!}-1}{x}$$

$$=\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n)!}-\frac{1}{x}$$

$$\int\frac{\cos x-1}{x}dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\int x^{2n-1}dx-\int\frac{1}{x}dx$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)(2n)!}-\ln x$$

In each argument, x is a constant and so, $\ln x$ can be treated as the integral constant.

Therefore $\int\frac{\cos x-1}{x}dx=\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)(2n)!}+C}$

Q50E

Consider the following indefinite integral:

$$\int \arctan\left(x^2\right)dx$$

Recall the Maclaurin series.

$$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Replace x with x^2 in the series.

$$\tan^{-1}\left(x^2\right)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$$

Apply the indefinite integral on both sides to solve the integral.

$$\int \tan^{-1}\left(x^2\right)dx=\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}dx$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int x^{4n+2}dx$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{x^{4n+3}}{4n+3}+C}$$, where C is an integration constant.

Q51E

Given that $f(x) = x^3(\arctan x)$.

We know that the Maclaurin expansion of $\arctan x$ is given by

$$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$$

Multiply both the sides by x^3.

$$x^3\tan^{-1}x = x^4-\frac{x^6}{3}+\frac{x^8}{5}-\frac{x^{10}}{7}+\dots$$

Integrate the expressions on both sides of the equation with respect to x.

$$\int x^3\tan^{-1}x\,dx = \int\left(x^4-\frac{x^6}{3}+\frac{x^8}{5}-\frac{x^{10}}{7}+\dots\right)dx$$

$$= C+\frac{x^5}{5}-\frac{x^7}{21}+\frac{x^9}{45}-\frac{x^{11}}{77}+\dots$$

Apply the limits.

$$\int_0^{1/2} x^3\tan^{-1}x\,dx = \left(C+\frac{x^5}{5}-\frac{x^7}{21}+\frac{x^9}{45}-\frac{x^{11}}{77}+\dots\right)_0^{1/2}$$

$$= \frac{(1/2)^5}{5}-\frac{(1/2)^7}{21}+\frac{(1/2)^9}{45}-\frac{(1/2)^{11}}{77}+\dots$$

$$\approx 0.00592-6.34131\times10^{-6}$$

$$\approx 0.0059$$

Therefore, $\int_0^{1/2} x^3\tan^{-1}x\,dx$ evaluates to $\boxed{0.0059}$.

Q52E

We have

$$\sin(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4(2n+1)}}{(2n+1)!}$$

$$\Rightarrow \int \sin(x^4)dx = \sum_{n=0}^{\infty} \int \left(\frac{(-1)^n x^{4(2n+1)}}{(2n+1)!} \right) dx$$

$$= \sum_{n=0}^{\infty} \int \left(\frac{(-1)^n x^{(8n+4)}}{(2n+1)!} \right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+5}}{(8n+5)(2n+1)!}$$

$$\Rightarrow \int_0^1 \sin(x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^{13}}{78} + \frac{x^{21}}{2520} - \dots \right]_0^1$$

$$= \left[\frac{1}{5} - \frac{1}{78} + \frac{1}{2520} - \dots \right]$$

$$= 0.2 - 0.01282 + 0.00039 - \dots$$

$$\cong 0.1879$$

By the Alternating Series Estimation Theorem we know that

$$|s - s_3| \le b_4 \text{ where } b_4 = \frac{1}{29 \cdot 7!}$$

$$= 0.00000684$$

So $\int_0^1 \sin(x^4)\,dx \cong 0.1879$ is corrected up to four decimal places.

Q53E

$\sqrt{1+x^4} = \left(1+x^4\right)^{1/2} = \sum_{n=0}^{\infty} \left(\frac{1/2}{n!} \right) \left(x^4\right)^n$, so:

$\int \sqrt{1+x^4}\ \mathrm{d}x = C + \sum_{n=0}^{\infty} \left(\frac{1/2}{n!} \right) \frac{x^{4n+1}}{4n+1}$ and hence, since 0.4<1, we have:

$$I = \int_0^{0.4} \sqrt{1+x^4}\ \mathrm{d}x = \sum_{n=0}^{\infty} \left(\frac{1/2}{n!} \right) \frac{(0.4)^{4n+1}}{4n+1}$$

$$= \frac{(0.4)^1}{0!} + \frac{1/2}{1!}\frac{(0.4)^5}{5} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{(0.4)^9}{9} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\frac{(0.4)^{13}}{13} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right.}{4!}$$

$$= 0.4 + \frac{(0.4)^5}{10} - \frac{(0.4)^9}{72} + \frac{(0.4)^{13}}{208} - \frac{5(0.4)^{17}}{2176} + \dots$$

Now $\frac{(0.4)^9}{72} \approx 3.6 \times 10^{-6} < 5 \times 10^{-6}$, so by Alternating Series Estimation Theorem,

$I \approx 0.4 + \frac{(0.4)^5}{10} \approx 0.40102$ (correct to 5 decimal places).

Q54E

We have to evaluate $\int_0^{0.5} x^2 e^{-x^2}\,dx \qquad (|\text{error}| < 0.001)$

Since the Maclaurin series for e^{-x^2} is

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots\dots\dots$$

Then $\quad x^2 e^{-x^2} = x^2 - \dfrac{x^4}{1!} + \dfrac{x^6}{2!} - \dfrac{x^8}{3!} + \dfrac{x^{10}}{4!} - \dots\dots\dots\dots$

$$\int x^2 e^{-x^2}\,dx = \int\left(x^2 - \frac{x^4}{1} + \frac{x^6}{2} - \frac{x^8}{6} + \frac{x^{10}}{24} - \dots\dots\right)dx$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54} + \frac{x^{11}}{264} - \dots\dots + C$$

Then $\quad \displaystyle\int_0^{0.5} x^2 e^{-x^2}\,dx = \left[\frac{(0.5)^3}{3} - \frac{(0.5)^5}{5} + \frac{(0.5)^7}{14} - \frac{(0.5)^9}{54} + \dots\dots\right]$

$$= 0.04166 - 0.00625 + 0.000558\dots\dots\dots$$

Since $\quad b_3 = 0.000558 = \dfrac{1}{1792} < \dfrac{1}{1000} = 0.001$

And $\quad S_2 = 0.03541$

By the Alternating series estimation theorem we know that

$$|s - s_2| \le b_3 < 0.001$$

This error does not affect the second decimal place, so we have

$$\boxed{I \approx 0.0354}$$

Q55E

We have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore

$$\lim_{n\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{n\to 0} \frac{x - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \dots}{x^2}$$

$$= \lim_{n\to 0}\left(\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} + \dots\right)$$

$$= \frac{1}{2}$$

Q56E

We have to evaluate $\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}$

$$\frac{1-\cos x}{1+x-e^x}=\frac{1-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots\ldots}{1+x-1-\frac{x}{1!}-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}\ldots\ldots}$$

$$=\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots\ldots\ldots\ldots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}-\ldots\ldots\ldots\ldots}$$

$$=\frac{1-\frac{x^2}{4\cdot 3}+\frac{x^4}{6\cdot 5\cdot 4\cdot 3}+\ldots\ldots}{-1-\frac{x}{3}-\frac{x^2}{4.3}-\ldots\ldots\ldots\ldots}$$

Then $\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}=\frac{1}{-1}$

$$=\boxed{-1}$$

Q57E

We have to evaluate $\lim_{x\to 0}\left(\frac{\sin x-x+\frac{1}{6}x^3}{x^5}\right)$

Use the Maclaurin series for sin x

$$\sin x-x+\frac{1}{6}x^3=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\ldots\right)-x+\frac{x^3}{6}$$

$$=\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\ldots\ldots\right)-x+\frac{x^3}{6}$$

$$=\frac{x^5}{120}-\frac{x^7}{5040}+\ldots\ldots$$

Then $\left(\sin x-x+\frac{1}{6}x^3\right)\frac{1}{x^5}=\frac{1}{120}-\frac{x^2}{5040}+\ldots\ldots\ldots$

And so $\lim_{x\to 0}\left(\sin x-x+\frac{1}{6}x^3\right)\frac{1}{x^5}=\lim_{x\to 0}\left[\frac{1}{120}-\frac{x^2}{5040}+\ldots\ldots\right]$

$$=\boxed{\frac{1}{120}}$$

Q58E

The series for $\tan x$ is given as

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots\ldots$$

Then $\frac{\tan x-x}{x^3}=\frac{1}{3}+\frac{2}{15}x^2+\ldots\ldots\ldots\ldots$

Therefore $\lim_{x\to 0}\frac{\tan x-x}{x^3}=\lim_{x\to 0}\left(\frac{1}{3}+\frac{2}{15}x^2+\ldots\ldots\ldots\right)$

$$=\boxed{\frac{1}{3}}$$

Q59E

We have to find the Maclaurin series for the function $y = e^{-x^2}\cos x$.

We know that the Maclaurin series for the function e^x is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Therefore $e^{-x^2} = \sum_{n=0}^{\infty}\frac{\left(-x^2\right)^n}{n!}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

$$= 1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+.........$$

Also the Maclaurin series for the function $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+.......$$

Therefore, the Maclaurin series for the product of e^{-x^2} and $\cos x$ is

$$e^{-x^2}\cos x = \left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+.....\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+.......\right)$$

$$= \left(1-\frac{x^2}{1}+\frac{x^4}{2}-......\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-........\right)$$

$$= 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^2}{1}+\frac{x^4}{2}-\frac{x^6}{24}+\frac{x^4}{2}-\frac{x^6}{4}+.......$$

$$= 1-\frac{x^2}{2}-x^2+\frac{x^4}{24}+\frac{x^4}{2}+\frac{x^4}{2}-.......$$

(Since we need only first three non zero terms)

$$= 1-\frac{3}{2}x^2+\frac{25}{24}x^4 ....$$

Hence

$$\boxed{e^{-x^2}\cos x = 1-\frac{3}{2}x^2+\frac{25}{24}x^4-......}$$

Q60E

$$y = \sec x = \frac{1}{\cos x}$$

$$= \frac{1}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+........} = \frac{1}{1-\frac{x^2}{2}+\frac{x^4}{24}-.......}$$

We use long division,

$$
\begin{array}{r|l}
 & 1+\dfrac{x^2}{2}+\dfrac{5}{24}x^4+\ldots\ldots\ldots\ldots\ldots \\
\hline
1-\dfrac{x^2}{2}+\dfrac{x^4}{24}-\dfrac{x^6}{720}+\ldots\ldots & 1 \\
 & 1-\dfrac{x^2}{2}+\dfrac{x^4}{24}-\dfrac{x^6}{720}\ldots\ldots \\
\cline{2-2}
 & \dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^6}{720}\ldots\ldots \\
 & \dfrac{x^2}{2}-\dfrac{x^4}{4}+\dfrac{x^6}{96}+\ldots\ldots \\
\cline{2-2}
 & \dfrac{5}{24}x^4+\ldots\ldots
\end{array}
$$

Then we have $y=\boxed{\sec x=1+\dfrac{x^2}{2}+\dfrac{5}{24}x^4+\ldots\ldots\ldots\ldots}$

Q61E

We have $y=\dfrac{x}{\sin x}$

The Maclaurin series for sin x is

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\ldots\ldots\ldots$$

Then $\dfrac{x}{\sin x}=\dfrac{x}{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots\ldots\ldots}$

We use a procedure like long division

$$
\begin{array}{r|l}
 & 1+\dfrac{x^2}{6}+\dfrac{7x^4}{360}+\ldots\ldots\ldots\ldots\ldots \\
\hline
x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\ldots\ldots & x \\
 & x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\ldots\ldots \\
\cline{2-2}
 & \dfrac{x^3}{6}-\dfrac{x^5}{120}+\ldots\ldots\ldots \\
 & \dfrac{x^3}{6}-\dfrac{x^5}{36}+\dfrac{x^7}{720}+\ldots\ldots\ldots \\
\cline{2-2}
 & \dfrac{7x^5}{360}-\dfrac{x^7}{720}+\ldots\ldots \\
 & \dfrac{7x^5}{360}-\dfrac{7x^7}{2160}+\ldots \\
\cline{2-2}
 & \dfrac{x^7}{540}-\ldots\ldots
\end{array}
$$

Thus $\boxed{\dfrac{x}{\sin x}=1+\dfrac{x^2}{6}+\dfrac{7}{360}x^4+\ldots\ldots\ldots\ldots\ldots}$

Q62E

We have

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$$

Now

$$e^x\ln(1+x) = \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials.

$$\begin{array}{rl} & 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \\ \times & x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \\ \hline & x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \\ + & x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots \\ & +\frac{x^3}{2} \\ \hline & x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\cdots \end{array}$$

$$\therefore e^x\ln(1+x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\cdots$$

Q63E

We have to find the sum of the series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}$

$$=\sum_{n=0}^{\infty}\frac{\left(-x^4\right)^n}{n!}$$

Compare this series with $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

We have $\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}=e^{-x^4}}$

Q64E

We have to find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{6}\right)^{2n}$$

Compare it with $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Then $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}}$

Q65E

Given series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n5^n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{3}{5}\right)^n}{n}$$

$$= \ln\left(1+\frac{3}{5}\right)$$

$$= \ln\left(\frac{8}{5}\right) \qquad \left(\because \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\right)$$

Q66E

We have to find the sum of the series $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!}$

Compare it with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Then $\boxed{\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = e^{3/5}}$

Q67E

We have to find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1}$$

Compare it with $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Then $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sin\left(\pi/4\right)$

$$= \frac{1}{\sqrt{2}}$$

Q68E

We have to find the sum of the series $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \ldots\ldots\ldots\ldots$

We know that

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

Then put x =ln 2, we get

$$e^{-\ln 2} = 1 - \frac{\ln 2}{1!} + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \ldots\ldots\ldots\ldots$$

Then $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \ldots\ldots\ldots = e^{-\ln 2}$

$$= e^{\ln(2)^{-1}}$$

$$= 2^{-1} = \boxed{\frac{1}{2}}$$

Q69E

We have to find the sum of the series

$$3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \ldots\ldots\ldots\ldots\ldots = \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \ldots\ldots\ldots\ldots$$

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

Then take x = 3

$$e^3 = 1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \ldots\ldots\ldots\ldots$$

Therefore $\frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \ldots\ldots\ldots\ldots = \boxed{e^3 - 1}$

Q70E

Given series

$$\frac{1}{1.2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \frac{1}{7.2^7} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{2n+1}$$

$$= \tan^{-1}\left(\frac{1}{2}\right) \qquad \left(\because \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\right)$$

$$\approx 0.46365$$

Q71E

Let $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots. + a_n x^4$

Then

$$p'(0) = a_1, p''(0) = 2!a_2 \ldots p^{(n)}(0) = n!a_n$$

Therefore the Maclamin series of P (x) is

$$P(x) = \sum_{i=0}^{n} \frac{p^{(i)}(0)}{i!} x^i$$

$$= \sum_{i=0}^{n} a_i x^i$$

Now $P(x+1)=\sum_{i=0}^{n} a_i (x+1)^i$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + \ldots + a_n(x+1)^n$$

$$= \left(a_0 + a_1 x + \ldots + a_n x^n\right) + \left(a_1 + 2a_2 x + \ldots + n a_n x^{n-1}\right)$$

$$+ \frac{\left(2a_2 + 6a_3 x + \ldots + n(n-1)a_n x^{n-2}\right)}{2!} + \ldots \frac{n! a_n}{n!}$$

$$= \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}$$

Q72E

Let $f(x) = \left(1+x^3\right)^{30}$

The Maclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

but binomial expresion of $\left(1+x^3\right)^{30}$ is

$$\left(1+x^3\right)^{30} = \sum_{n=0}^{30} c(30,n)\, x^{3n}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{30} c(30,n)\, x^{3n}$$

$$\Rightarrow f^{(58)}(0) = 0$$

(because in right hand side the coefficient of x^{58} is 0).

Q73E

It is given that $\left|f'''(x)\right| \le M$ for $|x-a| \le d$, which means that $f'''(x) \le M$ or $f'''(x) \ge -M$ for $a \le x \le d$

Let us consider $f'''(x) \le M$

So $\int_a^x f'''(t)\,dt \le \int_a^x M\,dt$

$$\Rightarrow f''(x) - f''(a) \le M(x-a)$$

$$\Rightarrow f''(x) \le f''(a) + M(x+a)$$

Integrating again

$$\Rightarrow \int_a^x f''(t)\,dt \le \int_a^x \left[f''(a) + M(t-a)\right]dt$$

$$\Rightarrow f'(x) - f'(a) \le f''(a).(x-a) + \frac{1}{2}M(x-a)^2$$

$$\Rightarrow f'(x) \le f'(a) + f''(a)(x-a) + \frac{1}{2}M(x-a)^2$$

Integrating again

$$\Rightarrow \int_a^x f'(t)dt \le \int_a^x \left[f'(a) + f''(a).(t-a) + \frac{1}{2}M(t-a)^2 \right] dt$$

$$\Rightarrow f(x) - f(a) \le f'(a).(x-a) + f''(a).\frac{1}{2}(x-a)^2 + \frac{1}{6}M(x-a)^3$$

$$\Rightarrow f(x) \le f(a) + (x-a).f'(a) + \frac{1}{2}(x-a)^2.f''(a) + \frac{1}{6}M(x-a)^3$$

$$\Rightarrow f(x) \le T_2(x) + \frac{1}{6}M(x-a)^3,$$

Where $T_2(x)$ is second degree Taylor polynomial

$$\Rightarrow f(x) - T_2(x) \le \frac{1}{6}M(x-a)^3$$

$$\Rightarrow R_2(x) \le \frac{1}{6}M(x-a)^3 \qquad \left[\text{since} R_2(x) = f(x) - T_2(x)\right]$$

Q74E

(A) $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replacing x by $\left(-1/x^2\right)$

We have $e^{-1/x^2} = \sum_{n=0}^{\infty} \frac{\left(-1/x^2\right)^n}{n!}$

$$f(x) = e^{-1/x^2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, x^{2n}}$$

We have $f(x) = 0$ for $x = 0$

Again we write $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\, x^{2n}}$

Taking limit as $x \to 0$, both sides of the equation

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\, x^{2n}}$$

Since $\lim_{x\to 0} f(x) = 0$ and $\sum_{n=0}^{\infty} \frac{(1-)^n}{(n!)\, x^{2n}} \to \infty$ $\quad$ *as* $x \to 0$

So both sides of the equation can not be equal

Therefore $\boxed{f(x) \ne \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\, x^{2n}}}$

(B) Now we graph the function, we see that near the origin the slope of the graph is very close to 0 or equal to 0.

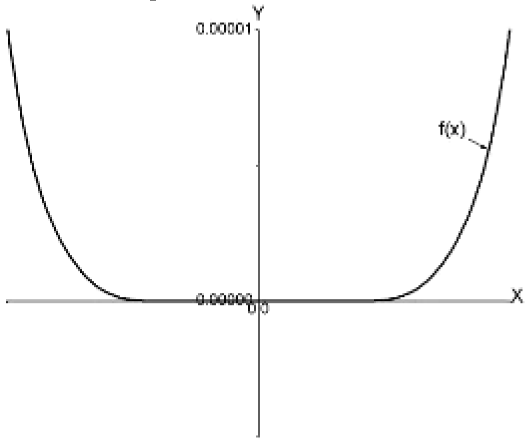


Fig.1

Q75E

(a)

Consider the binomial series: If k is any real number and $|x|<1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty}\binom{k}{n}x^n \\ &= 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots\end{aligned}$$

Let $g(x)=\sum_{n=0}^{\infty}\binom{k}{n}x^n$ and $g(x)=(1+x)^k \quad -1<x<1$

$$=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots$$

Differentiating with respect to x

$$\begin{aligned}g'(x) &= 0+k(1)+\frac{k(k-1)}{2!}(2x)+\frac{k(k-1)(k-2)}{3!}(3x^2)+\cdots \\ &= k+\frac{k(k-1)}{2!}(2x)+\frac{k(k-1)(k-2)}{3!}(3x^2)+\cdots \\ &= k\left(1+\frac{(k-1)}{2!}(2x)+\frac{(k-1)(k-2)}{3!}(3x^2)+\cdots\right)\end{aligned}$$

Since $g(x)=(1+x)^k$

$$\frac{g(x)}{(1+x)}=\frac{(1+x)^k}{(1+x)}$$ Dividing on both sides by $(1+x)$

$$\frac{kg(x)}{(1+x)}=\frac{k(1+x)^k}{(1+x)}$$ Multiplying on both sides by k

$$=k(1+x)^{k-1}$$

$$=k\left(1+(k-1)x+\frac{(k-1)(k-2)}{2!}x^2+\cdots\right)$$

$$=k\left(1+\frac{(k-1)}{2!}(2x)+\frac{(k-1)(k-2)}{2!\times 3}(3x^2)+\cdots\right)$$

$$=k\left(1+\frac{(k-1)}{2!}(2x)+\frac{(k-1)(k-2)}{3!}(3x^2)+\cdots\right)$$

$$=g'(x) \qquad -1<x<1$$

Therefore $g'(x)=\dfrac{kg(x)}{(1+x)} \quad -1<x<1$

(b)

Consider the binomial series: If k is any real number and $|x|<1$, then

$$(1+x)^{-k}=1-kx+\frac{k(k-1)}{2!}x^2-\frac{k(k-1)(k-2)}{3!}x^3+\cdots$$

Let $h(x)=(1+x)^{-k}$ and $g(x)=(1+x)^k$

Take, $h(x)=(1+x)^{-k}(1+x)^k$

$$\begin{aligned}h(x)&=(1+x)^{-k}(1+x)^k\\&=(1+x)^{-k+k}\\&=(1+x)^0\\&=1\end{aligned}$$

Differentiating with respect to x

Therefore, $h'(x)=0$

(c)

The traditional notation for the coefficient in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

If k is any real number and $|x|<1$, then

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$
$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The above series is converge when $|x|<1$

The above series whether or not it converges at the endpoint, ± 1, depends on the value of k

The series converges at 1 if $-1<k\le 0$ and at both endpoint if $k \ge 0$.

If k is a positive integer and $n>k$, then the expression for $\binom{k}{n}$ contain a factor $(k-k)$

So, $\binom{k}{n}=0$ for $n>k$

This means that the series $g(x)=(1+x)^k$

(c)

The traditional notation for the coefficient in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

If k is any real number and $|x|<1$, then

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$
$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The above series is converge when $|x|<1$

The above series whether or not it converges at the endpoint, ± 1, depends on the value of k

The series converges at 1 if $-1<k\le 0$ and at both endpoint if $k \ge 0$.

If k is a positive integer and $n>k$, then the expression for $\binom{k}{n}$ contain a factor $(k-k)$

So, $\binom{k}{n}=0$ for $n>k$

This means that the series $g(x)=(1+x)^k$

Q76E

By Exercise 25, $\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!}$, so we have

$$\left(1-x^2\right)^{1/2} = 1 - \frac{1}{2}x^2 - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n} \text{ and}$$

$$\sqrt{1-e^2\sin^2\theta} = 1 - \frac{1}{2}e^2\sin^2\theta - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}e^{2n}\sin^{2n}\theta$$

Therefore, we now find that

$$L = 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta = 4a\int_0^{\pi/2}\left(1 - \frac{1}{2}e^2\sin^2\theta - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}e^2\right.$$

$$= 4a\left[\frac{\pi}{2} - \frac{e^2}{2}S_1 - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n S_n\right]$$

where $S_n = \int_0^{\pi/2}\sin^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\frac{\pi}{2}$.

$$L = 4a\left(\frac{\pi}{2}\right)\left[1 - \frac{e^2}{2}\cdot\frac{1}{2} - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty} \frac{e^{2n}}{2^n}\cdot\frac{1^2\cdot 3^2\cdot 5^2\cdot\cdots\cdot(2n-3)^2(2n-1)}{n!\cdot 2^n\cdot n!}\right]$$

$$L = 4a\left(\frac{\pi}{2}\right)\left[1 - \frac{e^2}{2}\cdot\frac{1}{2} - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty} \frac{e^{2n}}{2^n}\cdot\frac{1^2\cdot 3^2\cdot 5^2\cdot\cdots\cdot(2n-3)^2(2n-1)}{n!\cdot 2^n\cdot n!}\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty} \frac{e^{2n}}{4^n}\left(\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{n!}\right)^2(2n-1)\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256} - \cdots\right]$$

$$= \frac{\pi a}{128}\left(256 - 64e^2 - 12e^4 - 5e^6 - \cdots\right)$$